# Writing a book with Barry Mazur "Prime Numbers and the Riemann Hypothesis" 

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SageMath, Inc. and University of Washington
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## Abstract

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In 2005, Barry Mazur and I started a decade+ project to write the book "Prime Numbers and the Riemann Hypothesis". This talk is about what's in the book and why, and how the book was produced.

Thank you to Barry and the organizers!
Encourage people to rudely interrupt me during my talk and ask questions!

## Prelude: collaborate with great co-authors!

Writing John Tate's Galois Cohomology notes for PCMI 1999...
> "Everybody is so jealous of you getting to talk with John Tate!"
> - David Savitt

If you ever get the chance to write something with someone incredible, take it!!
(I next wrote a long paper with Ken Ribet from that same PCMI.)

## Overview

(1) §1. Barry's Public Lecture
(2) §2. Writing a Book
(3) §3. Publishing a Book

# §1. Barry's Public Lecture 



## Clay Math Institute public lecture (MIT, May 3, 2005)

"Are there still unsolved problems about the numbers 1, 2, 3, 4, ... ?"

Use primes to "sell" number theory to the general public

- Immediately accessible
- Immediately interesting
- How erratic primes are
- Cicada's every 13,17 years...
- Many examples of "open, interesting questions"
- People can immediately make computations of their own
- Barry got his father, who had done NO math, hooked on the Goldbach Conjecture, so thought primes would work.


## SageMath



## I also launched SageMath in 2005

I launched Sage a few months before this 2005 CMI public lecture.

- Sage is a free open source alternative to Mathematica, Maple, Magma, and Matlab.
- Early Sage development motivated by this talk
- Linking Sage to Mathematica to compute 「
- Early visualization functionality
- Prime enumeration (via PARI)


## More about what was in Barry's public lecture...

## Topics

- Primes as atoms: integer factorization
- The largest known prime
- Enumerating primes: Sieve of Eratosthenes
- Twin primes
- Counting primes
- Gauss's Conjecture: The Prime Number Theorem
- Riemann: Fourier style smooth approximations $R_{k}(x)$ to $\pi(x)$
- Riemann's Harmonics: zeros of $\zeta(s)$


## It worked!

## §2. Writing a Book

## "Let's write a book..." - Barry

Could we turn this public lecture into a "popular book"?

- Write something for a general audience
- Small and readable
- Full of mathematics, not stories of people
- Profusely illustrated
- Meet for a few weeks in his country house and focus on this


## What kind of book?

There are already 4 recent popular books on the Riemann Hypothesis. Why write another?

## Our book could be unique

- Motivate by connecting the prime counting problem with our other research on the explicit formula
- Mostly math and not "stories of people" (other books on RH already do the stories well)


## What Sort of Book: Small, Medium or Large?

## Like T. C. MITS or like ON BULLSHIT?



## Our Approach

Go back 150+ years and explain what RH is more from the point of view of classical Fourier analysis.

- Embrace a mid-19th century very $\mathbb{R e a l}$ perspective
- Leave Complex numbers to the very, very end



## Target Audience?

## Who are we writing this book for?

Lovers of number theory who want to read about mathematics.

- Bright high school students? ${ }^{a}$
- Retired electrical engineers?

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## SageMath again

## Computations with Sage drove the exposition

We used Sage to compute with prime numbers, zeros, etc., and generally to plot everything in the book.

- Numerous plots that are absolutely essential to the exposition, and in fact really drove it!
- Surprising to see so much with such little computation.
- Central hook of the book appears from computation: "Fourier transform links the discrete distribution at prime powers and the discrete distribution of zeros of $\zeta(s)$."
- This is also what got us thinking about "how explicit is the explicit formula?" (another research project...)


## Collaborative $\operatorname{AT} T_{E X}$ via CoCalc

## How we wrote the book

- I wrote CoCalc's $\mathrm{A}^{\mathrm{A}} \mathrm{E}_{\mathrm{E}} \mathrm{X}$ editor for this book project:
- In a web browser
- Both of us simultaneously editing the same file
- Precise history of all changes
- Gives a sense of the collaborative spirit
- Rough PDF of book on the web at every stage
- GitHub tracking of changes
- Sage computations run in the same place as editing book
- Barry very closely read and understood the Sage code


## CoCalc's Collaborative LTTEX Editor



## CoCalc's Collaborative ATEX Editor and Sage Worksheet



# Here is the book! 

(in just a few slides)

## The Prime Counting Problem

Let $\pi(x)$ be the number of primes $\leq x$.
Problem: Give a "good approximation" for $\pi(x)$.
plot(prime_pi, 0, 100, color='red', figsize=[8,4])


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```
plot(prime_pi, 0, 1000, color='red', figsize=[8,4])
```



## Focus on The Prime Counting Problem

Let $\pi(x)$ be the number of primes $\leq x$.
Problem: Give a "good approximation" for $\pi(x)$.


## Answer: The Riemann Hypothesis (first formulation)

$$
\operatorname{Li}(X)=\int_{2}^{X} \frac{1}{\log (t)} d t \text { is good approx to } \pi(X)=\#\{\text { primes } \leq X\}
$$

$$
\text { For } X=10^{24}
$$



Figure 13.1: Plots of $\operatorname{Li}(X)$ (top), $\pi(X)$ (in the middle), and $X / \log (X)$

$$
\begin{aligned}
\pi(X) & =18,435,599,767,349,200,867,866 \\
\operatorname{Li}(X) & =18,435,599,767,366,347,775,143.10580 \ldots \\
X /(\log (X)-1) & =18,429,088,896,563,917,716,962.93869 \ldots \\
\operatorname{Li}(X)-\pi(X) & =r \\
\sqrt{X} \cdot \log (X) & =r 17,146,907,277.105803 \ldots \\
& 55,262,042,231,857.096416 \ldots
\end{aligned}
$$

RH1: The number of prime numbers less than $X$ is approximately $\mathrm{Li}(X)$ and this approximation is essentially square root accurate.

Gauss wrote in his 1849 letter that there are 216,745 prime numbers less than three million. This is wrong: the actual number of these primes is 216,816 .

## Answer: The Riemann Hypothesis (second formulation)

$\psi(x)$ : "A new staircase that starts on the ground at $x=0$ and the height of the riser of the step at $x=1$ will be $\log (2 \pi)$. The height of the riser of the step at $x=p^{n}$ will not be 1 but rather: the step at $x=p^{n}$ will have the height of its riser equal to $\log p$."


RH2: The prime power staircase $\psi(X)$ is essentially square root close to the 45 degree straight line.

## Answer: The Riemann Hypothesis (third formulation)

We deleted this formulation from the book, since it was too technical to state properly (it's the explicit formula). After deleting this, we accidentally didn't relabel the "fourth formulation", which confused readers.

Instead, we illustrate the heck out of it!

RH3: The Fourier transform of $\psi^{\prime}(X)$ "is basically" a discrete distribution supported at the imaginary parts of the (nontrivial) zeros of $\zeta(s)$.

## Fourier transform of $\Psi^{\prime}(x)$ (just four terms!)

$$
\begin{aligned}
f(t)=- & \frac{\log (2)}{2^{1 / 2}} \cos (t \log (2))-\frac{\log (3)}{3^{1 / 2}} \cos (t \log (3)) \\
& \quad-\frac{\log (2)}{4^{1 / 2}} \cos (t \log (4))-\frac{\log (5)}{5^{1 / 2}} \cos (t \log (5))
\end{aligned}
$$



- Anybody can easily plot this.
- Arrows point to imaginary parts of zeros of $\zeta(s)$ !


## Fourier transform of $\Psi^{\prime}(x)$ (first 20 terms)



$$
-\sum_{p^{n} \leq 20} \frac{\log (p)}{p^{n / 2}} \cos \left(t \log \left(p^{n}\right)\right)
$$

## Fourier transform of $\Psi^{\prime}(x)$ (first 500 terms)



$$
-\sum_{p^{n} \leq 500} \frac{\log (p)}{p^{n / 2}} \cos \left(t \log \left(p^{n}\right)\right)
$$

Take this home: The Fourier transform of the derivative of the prime power staircase "is" the zeros of the Riemann zeta function.

## It goes both ways!

The Fourier transform of the zeros "is" prime powers:


$$
-\sum_{i=1}^{1000} \cos \left(\log (s) \theta_{i}\right)
$$

## Riemann untangled this to get $\pi(x) \ldots$

We finish book with manipulation to approximate $\pi(x)$ by a sum of smooth functions $R_{k}(x)$ involving the $\theta_{i}$.


Figure 36.8: The function $R_{25}$ approximating the staircase of primes up to 100 Inspiration: Zagier's lecture "The First 50 million prime numbers".

## $R_{50}$ approximates $\pi(x)$ very well!



Figure 36.11: The function $\operatorname{Li}(X)$ (top, green), the function $R_{50}(X)$ (in blue), and the staircase of primes on the interval from 350 to 400.

## Answer: The Riemann Hypothesis (fourth formulation)

RH4: All the nontrivial zeroes of $\zeta(s)$ lie on the vertical line in the complex plane consisting of the complex numbers with real part equal to $1 / 2$.

## §3. Publishing a Book

## How to Publish the book: Self publish!?

## Self publishing?

Just put it on my website and see what happens.

- Some people read it...
- It didn't really get significant traction.
- There was still that key missing quality step.

Will Hearst convinced us to publish with a commercial publisher. Maybe he was tired of printing out copies to give to people?


## Finding a publisher

## Finding the right publisher for this book...

- Barry and I have both published a few books with a couple of publishers, over the years.
- Talked to many editors (the JMM was very helpful!)
- Looked at reputation, similar books, and who followed up
- Balanced competing goals (e.g., price, quality, rights)
- Kaitlin Leach from Cambridge University Press won.


## Typos and Mistakes

Or, making the book easier for people to read

- Dozens of people carefully read drafts of the book and provided incredibly useful feedback. THANK YOU!!
- The publisher had a copy editor read the book, and provided complementary feedback.
- Don't expect your publisher to catch the sort of mistakes a mathematician would catch (should be 1777-1855):



## Creating a Cover

## Ideas for Components Included...

- Title of book
- Our names
- Plot of $\zeta(s)$, using Sage's complex_plot
- Portrait of Riemann, the star of the book!
- Plots illustrating the main ideas of the book
- A "classical" look

There is a natural tension here: publisher vs author vs marketer

## What We Created



The Actual Cover


## Endorsements for the back cover

## Will Hearst and David Mumford kindly wrote about our book...



## Production

Producing the book

- Initial friction with production, e.g., "Please provide Microsoft Word document." (Cambridge Univ Press has made many positive steps toward better ${ }^{\mathrm{LT}} \mathrm{T}_{\mathrm{E}} \mathrm{X}$ support.)
- An unfortunate physical issue with some of the first printing.
- CUP strongly supported and marketed the book.
- Working with CUP has been a very positive experience overall.


## Published!



Kindle Store , Kindle eBooks s Science \& Math

## Prime Numbers and the Riemann Hypothesis 1st Edition, Kindle Edition

by Barry Mazur (Author), William Stein (Author)

## Prime Numbers and the Ricmann Hypothesis

## User Reviews

## Top customer reviews

Oliver

## Whort，gorgeous，inspring and affordable．

June 27， 2016
Format：Paperback Verified Purchase
A couple of books on the Riemann hypothesis have appeared for the general public：Derbeshire 2003，Du Sautoiy 2003， Sabbagh 2003，Rockmore 2005，Watkins 2015，van der Veen and van der Craats 2015 and now Mazur－Stein 2016．More for mathematicians are Koblitz 1977，Edwards 2001，and Stopple 2003．From general expositions，one should also mention the paper of Conrey of 2003 which won the Conant prize for expository writing as well as a nice paper of Bombieri of 1992．Is this too much for the subject？No．A problem like the Riemann hypothesis can never be written too much about，especially if texts are written by experts．It is the open problems which drive mathematics．The Riemann hypothesis is the most urgent of all the open problems in math and like a good wine，the problem has become more valuable over time．What helped also is that since the time of Riemann，more and more connections with other fields of mathematics have emerged．The book of Veen－Craats and Mazur－Stein have emerged about at the same time．They are both small and well structured．Veen－Craats has been field tested with high school students and has focus mostly on the gorgeous Mangoldt explicit formula for the Chebychev prime distribution function，sometimes called the＂music of the primes＂．Mazur－Stein do it similarly，however stress more on the Riemann spectrum and go didactically rather gently into the mathematics of Fourier theory as well as the theory of distributions．The book is carefully typeset，has color prints and some computer code for Sage．While Veen－Craats has many nice exercises，an exercise of Mazur－Stein led me to abandon other things for a couple of weeks，since it was so captivating．So be careful！A student who has taken basic calculus courses，should be able to read it．By the way，except Sabagh＇s book＂Dr Riemann＇s zeros＂，which was written by a writer and journalist，the other books were created by professional mathematicians．The Mazur－Stein book has probably the best＂street cred＂among the RH books for the general audience：both have done important work in number theory，also related to zeta functions：Mazur＇s name is on one of the grand generalizations of the Riemann zeta

discussion of the Reimann spec Some good graphics and a good Reimann spectrum．I found Derb Obsession＂overall more interest for the non－expert．
Published 1 year ago


Ricardo J．Menéndez
気领岺 Five Stars
Excellent presentation！Ricardo Published 1 year ago


Saul

## 用解 Very Good for Ge

## Some Math Background

Very informative．Not for the pre about it already，of course．
Published 1 year ago

david bailey
用领 Four Stars
satisfactory
Published 1 year ago

## Magazine \& Blog Reviews

- Sarnak review in Bulletins: "make effective use of such technology and do a marvelous job of integrating all this information into an exposition of the underlying mathematics."
- Avner Ash in American Math Monthly
- MAA review: "This book is a splendid piece of work, informative and valuable."
- John Baez: "It's the best elementary introduction to the connection between prime numbers and zeros of the Riemann zeta function."
- Cathy O'Neil: "If I have one complaint it's all the pictures of white male mathematicians. [...] it would have been even better if it focused on the ideas more and the people less."


## \$ Royalties \$

We sold some copies, so Cambridge University Press sent us some money. I'm spending my share on expenses for my dream dog:


## Translations

## 25 Apr 2018

Dear Professor Stein,

Prime Numbers and the Riemann Hypothesis

I am delighted to inform you that we are currently concluding an agreement with Nippon Hyoron Sha for a Japanese language edition of your book. They plan to print an edition of 2,500 copies initially, which will be sold at approximately 2,200 JPY per copy.
§3. Publishing a Book

## Future plans

## Someday we hope to...

- Create online fully interactive version of all the plots, which don't require knowing Sage to use.
- Finish related research on L-series of elliptic curves, connecting the rank to statistical behavior of the $a_{p}$.

Thank You!


[^0]:    ${ }^{a}$ Neither Barry nor I graduated from high school.

