Hidden Spectral Symmetries and Mode Stability of Rotating Black Holes

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Outline

1. BH Perturbations

2. Stability properties of Kerr

3. Stability properties of Kerr-de Sitter

4. Conclusions

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2. Stability properties of Kerr

3. Stability properties of Kerr-de Sitter

4. Conclusions

Kerr Black Holes

Astrophysical BHs are believed to be described by the Kerr metric:

$$ds^2 = -\frac{\Delta}{\Sigma}\left(dt^2 - a\sin^2\theta d\varphi\right)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left(\left(r^2+a^2\right)d\varphi - adt\right)^2$$

$$\Delta = (r-r_+)(r-r_-) \qquad\qquad \Sigma \equiv r^2 + a^2\cos^2\theta$$

Event/Cauchy horizon: $r_\pm = M \pm \sqrt{M^2 - a^2}$

mass

angular momentum per unit mass

Maximally-rotating (extremal) Kerr is for $a = M$

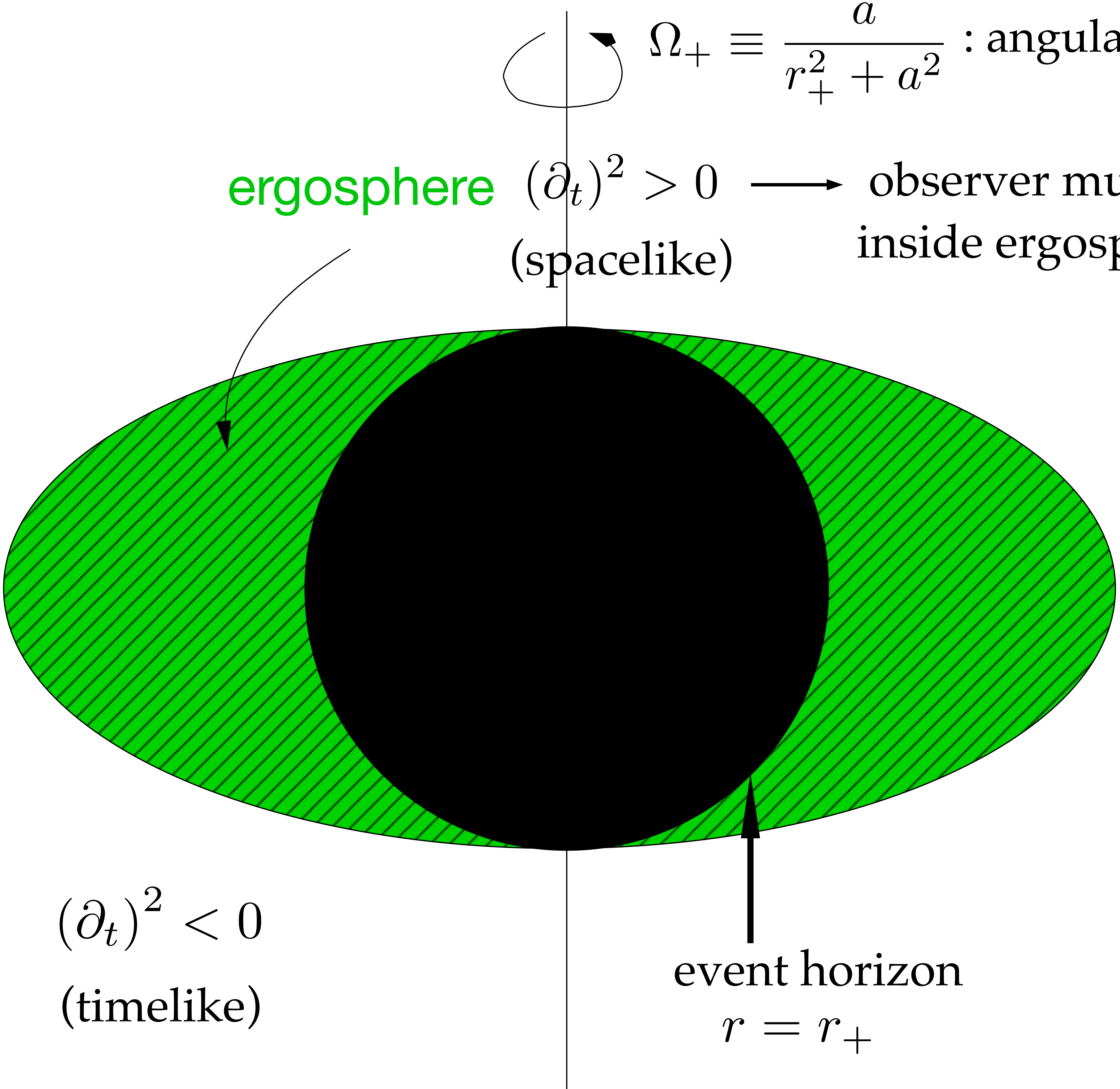
Horizon at $r = r_\pm$ has angular veloc. $\Omega_\pm \equiv \dfrac{a}{r_\pm^2 + a^2}$

and surface gravity $\kappa_\pm = \dfrac{r_+ - r_-}{2\left(r_\pm^2 + a^2\right)}$

It has a curvature singularity at $r = 0$ and $\theta = \pi/2$

It has two *explicit* symmetries (Killing vectors):

stationarity (∂_t) and axi-symmetry (∂_φ)



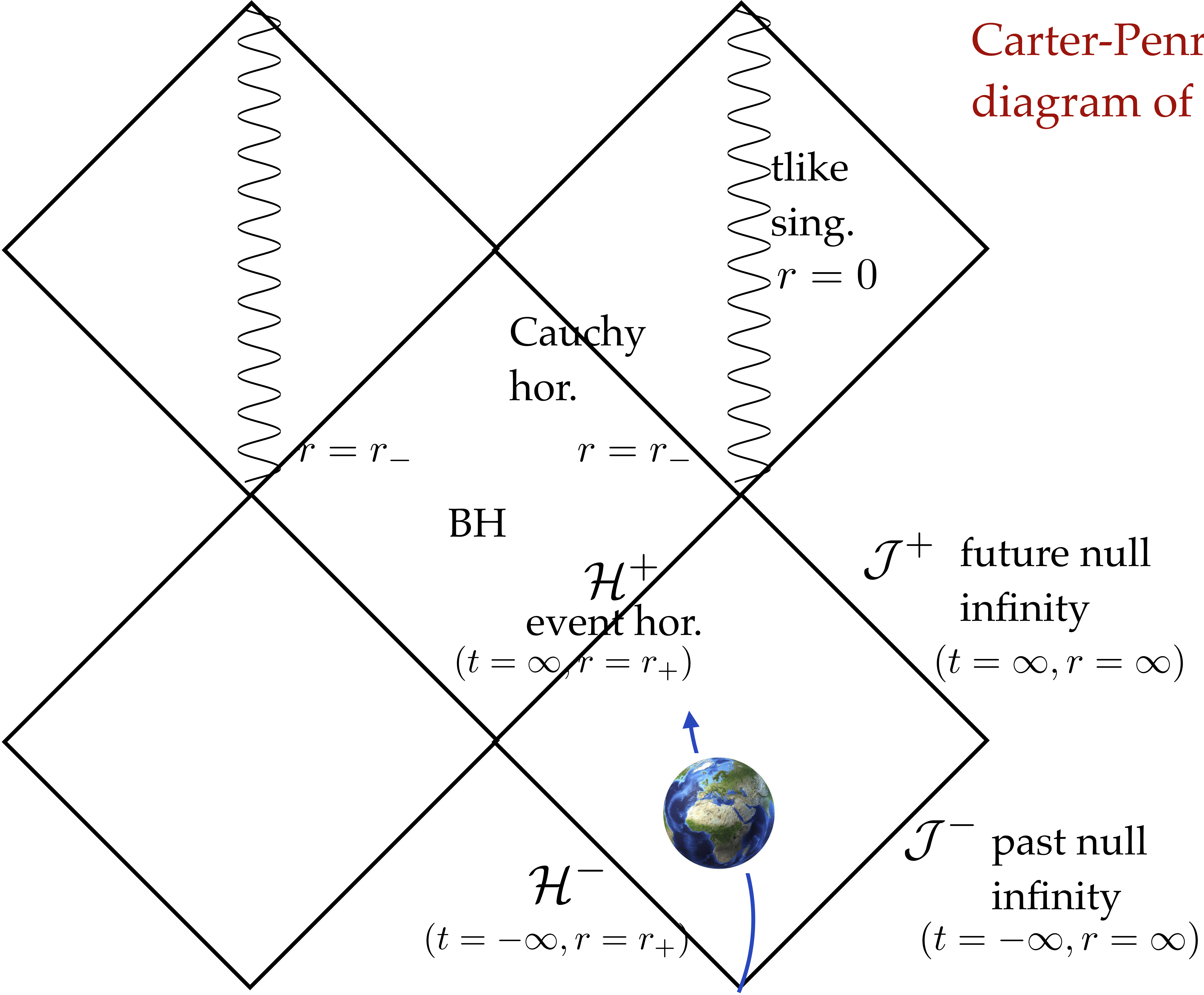
$\Omega_+ \equiv \dfrac{a}{r_+^2 + a^2}$: angular velocity

ergosphere $(\partial_t)^2 > 0$ (spacelike) $\longrightarrow$ observer must rotate inside ergosphere

$(\partial_t)^2 < 0$ (timelike)

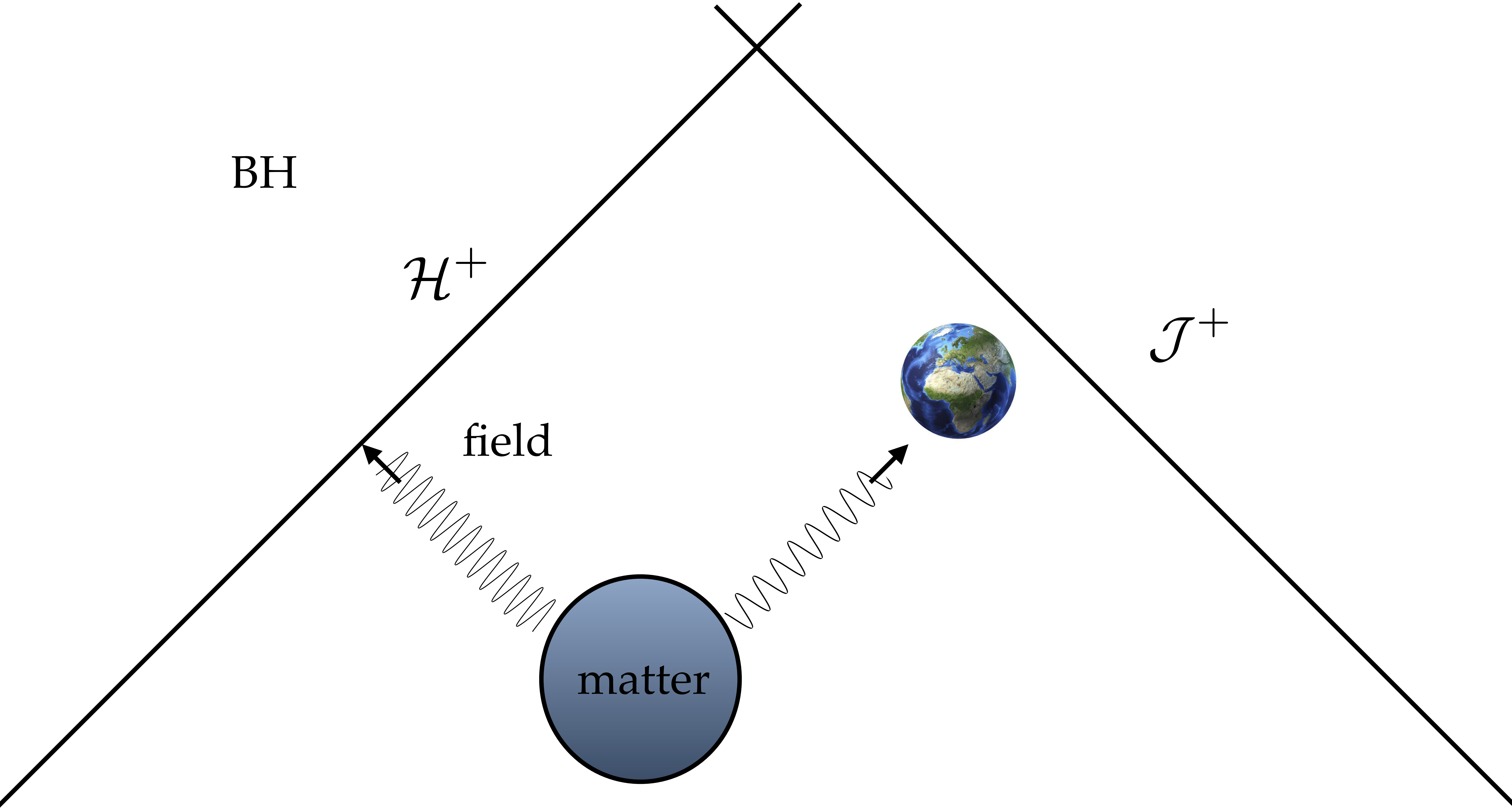
event horizon $r = r_+$

Carter-Penrose diagram of Kerr



BH Perturbations

BHs are not in isolation but are ‘perturbed’ by fields (scalar, fermion, electromagnetic, gravitational…) due to neighbouring matter (eg, accretion disk, neutron star, etc) or another BH



Important question in order to ascertain whether a Kerr BH is really the final stage of gravitational collapse of massive stars:

is Kerr spacetime stable under field perturbations?

Wave Equation

We consider linear field perturbations of a *fixed* BH (ie, we do not consider the backreaction of the field on the BH) $\longrightarrow$ the fields propagate on a BH *background* $g_{\mu\nu}$

E.g., scalar field perturbations ϕ of a BH satisfy a wave eq. (2nd order hyperbolic linear PDE)

$$\Box\phi(x) \equiv g_{\mu\nu}\nabla^{\mu}\nabla^{\nu}\phi(x) = T(x)$$: source of field

$$||$$

$$\frac{\partial}{\partial r}\left(\Delta\frac{\partial\phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \left(\frac{1}{\sin^2\theta} - \frac{a^2}{\Delta}\right)\frac{\partial^2\phi}{\partial\varphi^2} -$$

$$\frac{4aMr}{\Delta}\frac{\partial^2\phi}{\partial t\partial\varphi} - \left(\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right)\frac{\partial^2\phi}{\partial t^2}$$

Perturbations by other fields satisfy a similar wave eq.

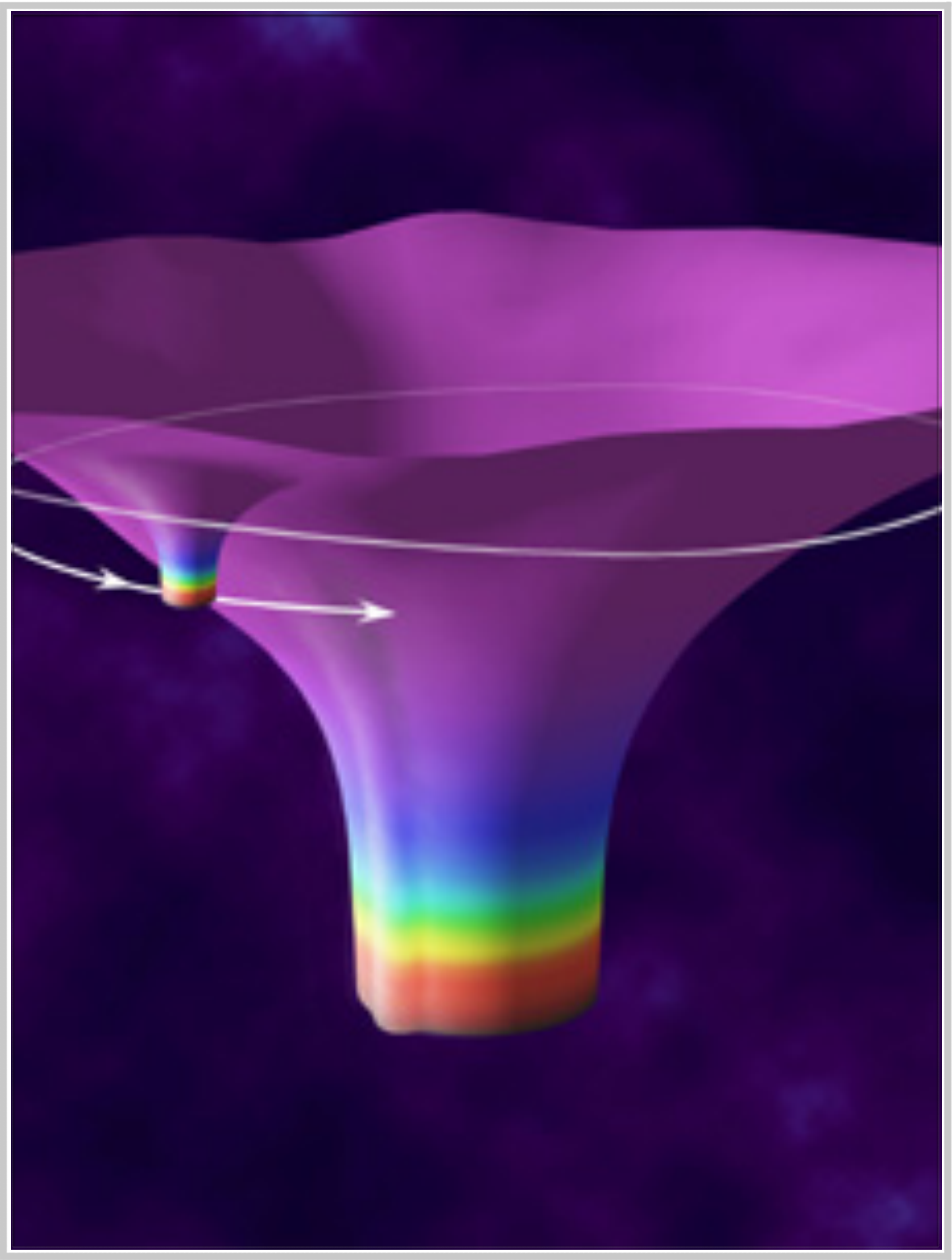
Eg, for grav. field perturbations, *linearize* Einstein eqs.

Smaller BH (m) moving on the background metric $g_{\mu\nu}$ of a massive BH (M) causes perturbation metric $h_{\mu\nu}$

due to M

$$\text{Total metric} \;= \overset{\downarrow}{g_{\mu\nu}} + \underset{\uparrow}{h_{\mu\nu}} + O\left(\frac{m}{M}\right)^2$$

perturbation (gravitational waves) due to m



Credit: NASA

The eqs. satisfied by the different components of $h_{\mu\nu}$ do not decouple, but…

Teukolsky'73 managed to decouple the eqs. satisfied for combinations ψ of different components and derivatives of the various fields (spin |s|=0 scalar, =1/2 neutrino, =1 emag for Faraday tensor, =2 grav for Weyl tensor)

Teukolsky "master" PDE: $\hat{\mathcal{O}}\psi(x) = T_s(x)$

↑ source of field

It's a wave-like eq.:

$$\hat{\mathcal{O}} \equiv \left(\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right)\frac{\partial^2}{\partial t^2} - 2s\left(\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right)\frac{\partial}{\partial t}$$

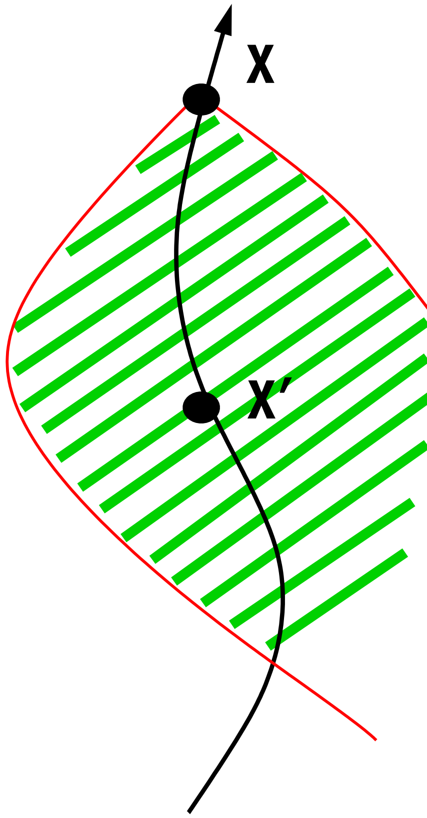
$$+\frac{4aMr}{\Delta}\frac{\partial^2}{\partial t\partial\varphi} + \left(\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right)\frac{\partial^2}{\partial\varphi^2} - 2s\left(\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right)\frac{\partial}{\partial\varphi}$$

$$-\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + (s^2\cot^2\theta - s)$$

Green Function

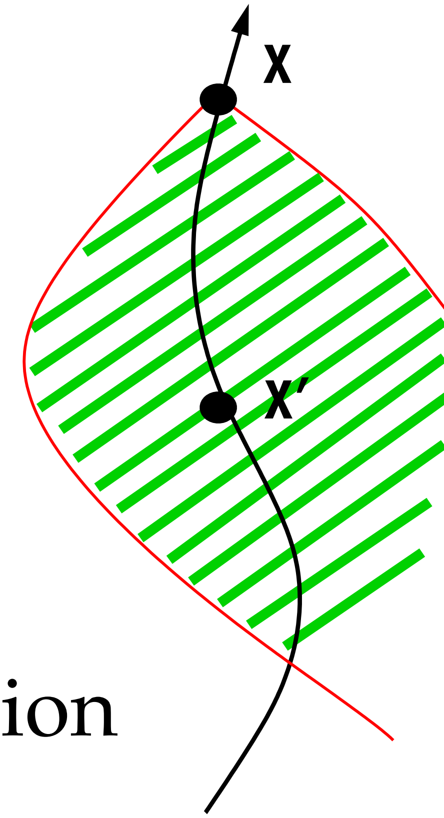
A crucial object is the retarded Green function

$\hat{\mathcal{O}}\, G_{ret}(x,x') = \delta_4(x,x')$ with causal b.c.:



$G_{ret}(x,x') = 0$ if $x' \neq J^-(x)$

Green Function



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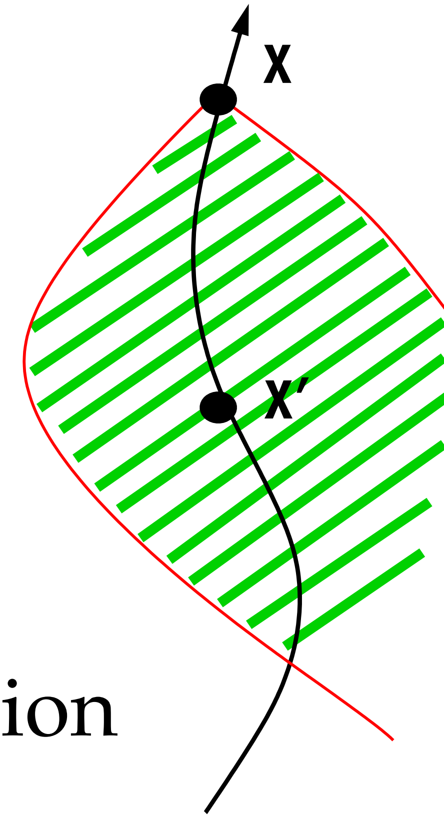
GF determines evolution in time of any initial field configuration

$$\psi(x) = \int_{t=0} \left[G_{ret}(x,x')\dot{\psi}^{ic}\,(\vec{x}') + \psi^{ic}\,(\vec{x}')\,\partial_t G_{ret}(x,x') \right] d^3\vec{x}'$$

Green Function



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GF can be calculated by *separating* (Carter’68) into Fourier modes (← stationarity) and spheroidal harmonics (← axisymmetry & *hidden* symmetry: Killing-Yano tensor) :

$$G_{ret} = \sum_{\ell,m}\int_{-\infty}^{\infty} d\omega\; e^{im\varphi - i\omega t} S_{\ell m\omega}(\theta) S_{\ell m\omega}(\theta') \overset{\uparrow\ \text{Fourier modes}}{G_{\ell m\omega}(r,r')}$$

The GF Fourier modes satisfy the radial Teukolsky eq.:

$$\hat{\mathcal{O}}_r \; G_{\ell m \omega}(r, r') = \delta(r, r')$$

2nd order linear operator in r

$$\hat{\mathcal{O}}_r \equiv \Delta \frac{d^2}{dr^2} + \frac{[\omega(r^2+a^2) - am - is(r-M)]^2}{\Delta} + \left(\frac{M^2 - a^2}{\Delta} + 4is\omega r - \lambda - a^2\omega^2 + 2am\omega \right)$$

separation const.: eigenvalue of angular ODE

(λ is invariant under $s \to -s$)

It can be trivially transformed to a confluent Heun ODE: 2 regular singular pts. at $r = r_\pm$ & 1 irregular singular pt. at $r = \infty$

The GF of a 2nd order linear ODE can be found from two linearly independent slns. of the *homogeneous* radial ODE:

$$G_{\ell m\omega}(r, r') = \frac{R^{in}_{\ell m\omega}(r_<)R^{up}_{\ell m\omega}(r_>)}{W}$$

Wronskian

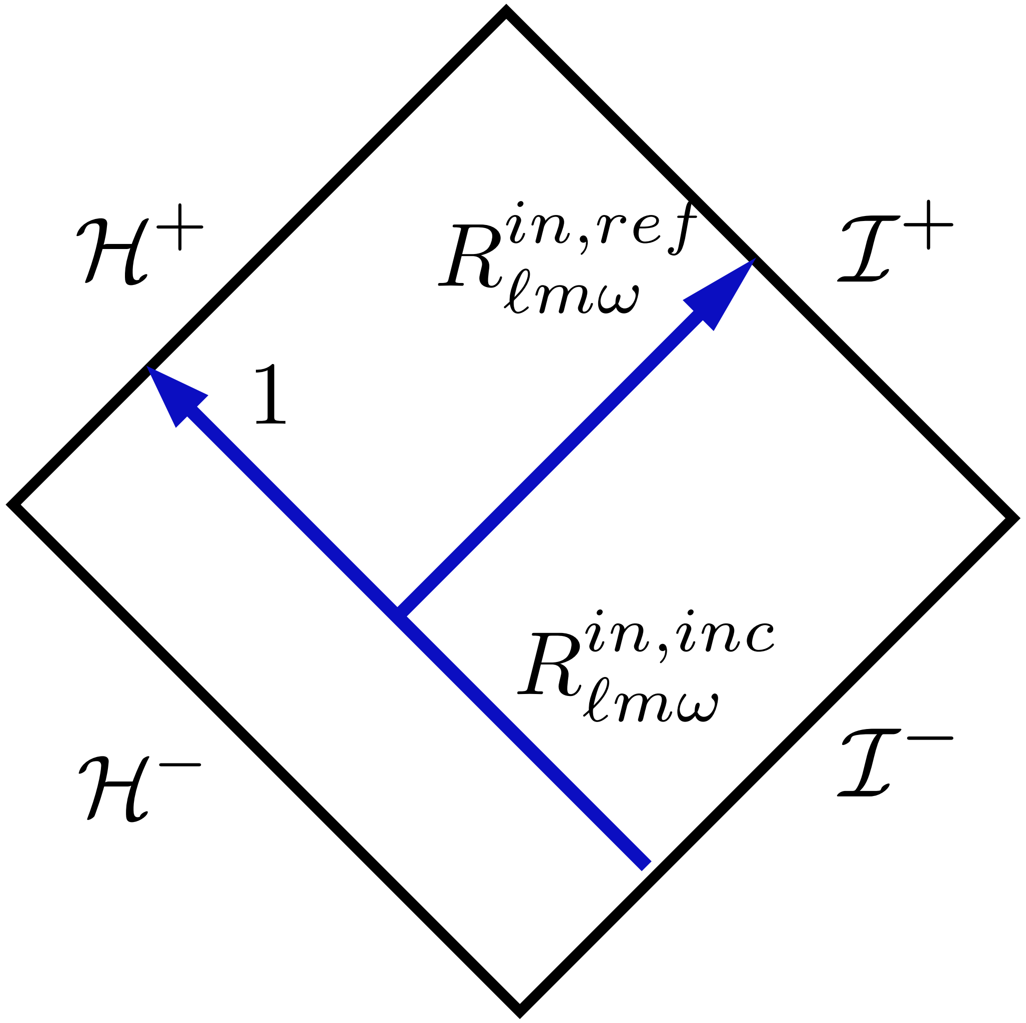
$$r_< \equiv \min{(r, r')}$$

$$r_> \equiv \max{(r, r')}$$

where $\hat{\mathcal{O}}_r \; R^{in/up}_{\ell m\omega}(r) = 0$

Causal boundary conditions for the homogeneous slns.:

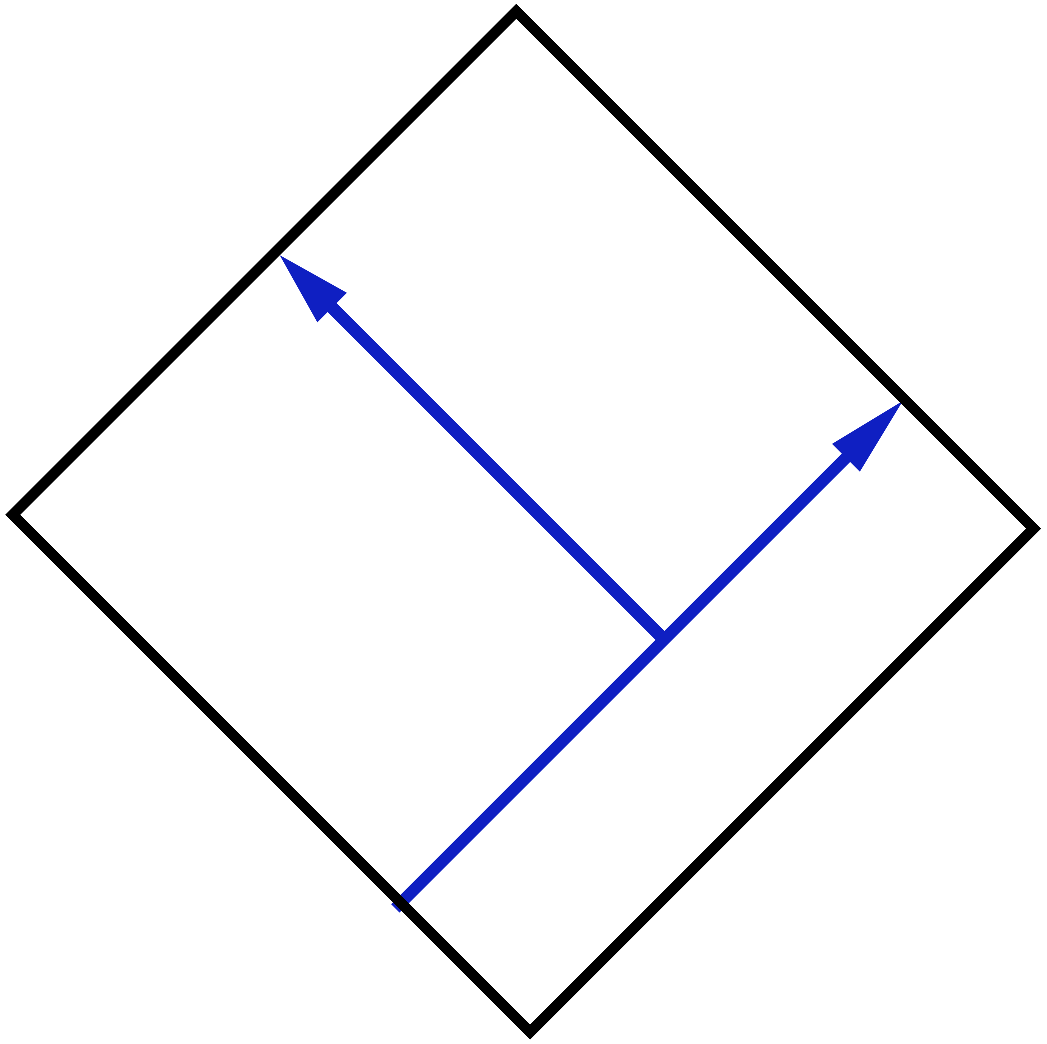
$$e^{-i\omega t} R^{in}_{\ell m\omega}(r)$$



$$R^{in}_{\ell m\omega} \sim e^{-i\omega_+ r_*}, \quad r_* \to -\infty$$
$$(r \to r_+)$$

is purely-ingoing into the horizon

$$e^{-i\omega t} R^{up}_{\ell m\omega}(r)$$



$$R^{up}_{\ell m\omega} \sim e^{+i\omega r_*}, \quad r_* \to +\infty$$
$$(r \to \infty)$$

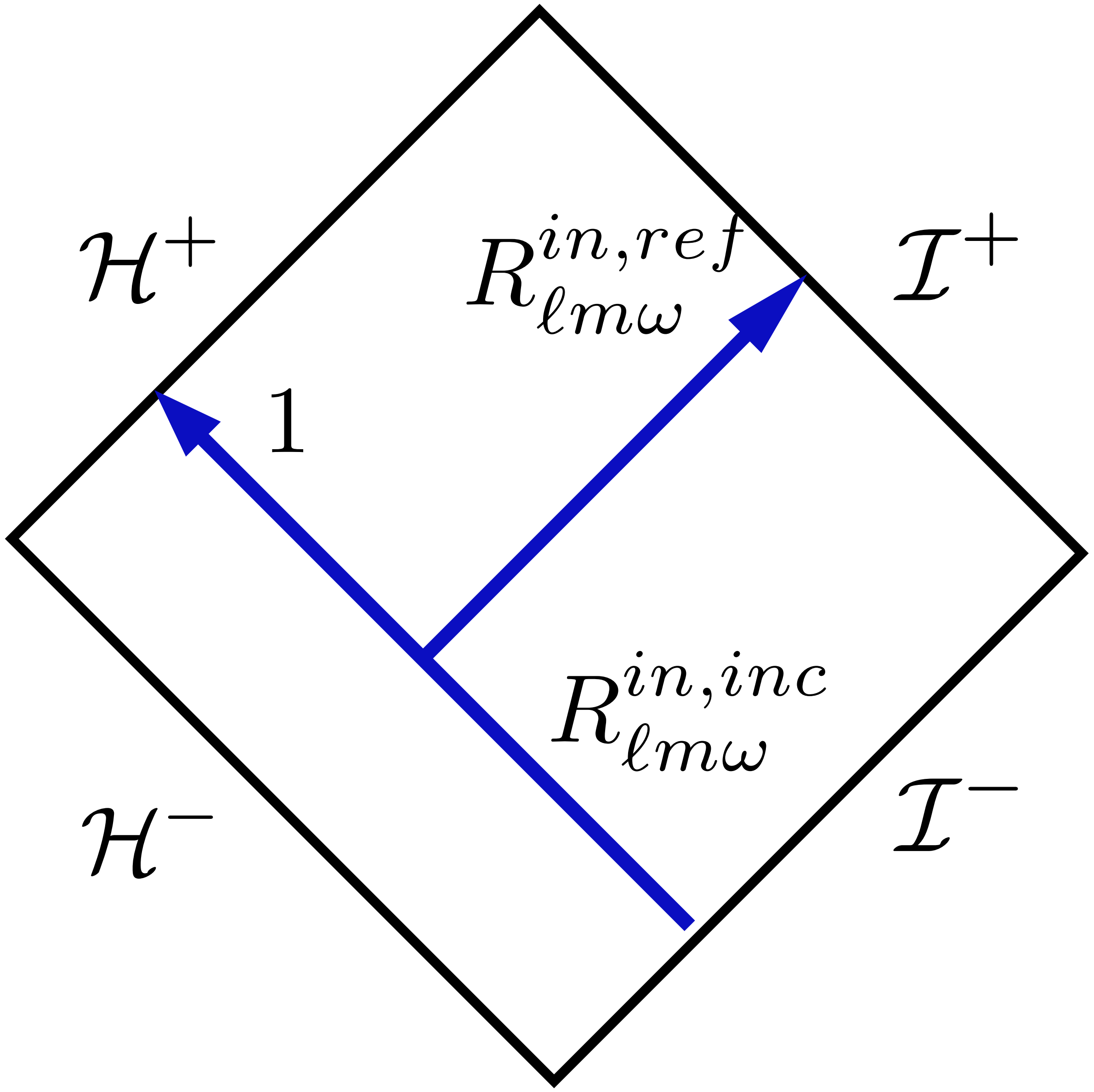
is purely-outgoing to infinity

$$W = 2i\omega R^{in,inc}_{\ell m\omega} \qquad \omega_\pm \equiv \omega - m\Omega_\pm \qquad \frac{dr_*}{dr} \equiv \frac{r^2+a^2}{\Delta}$$

Superradiance

Wronskian condition (energy conservation):

$$\left|R^{in,ref}_{\ell m\omega}\right|^2 = \left|R^{in,inc}_{\ell m\omega}\right|^2 - \frac{\omega_+}{\omega}$$



Superradiance: reflected wave has more energy than incident wave if

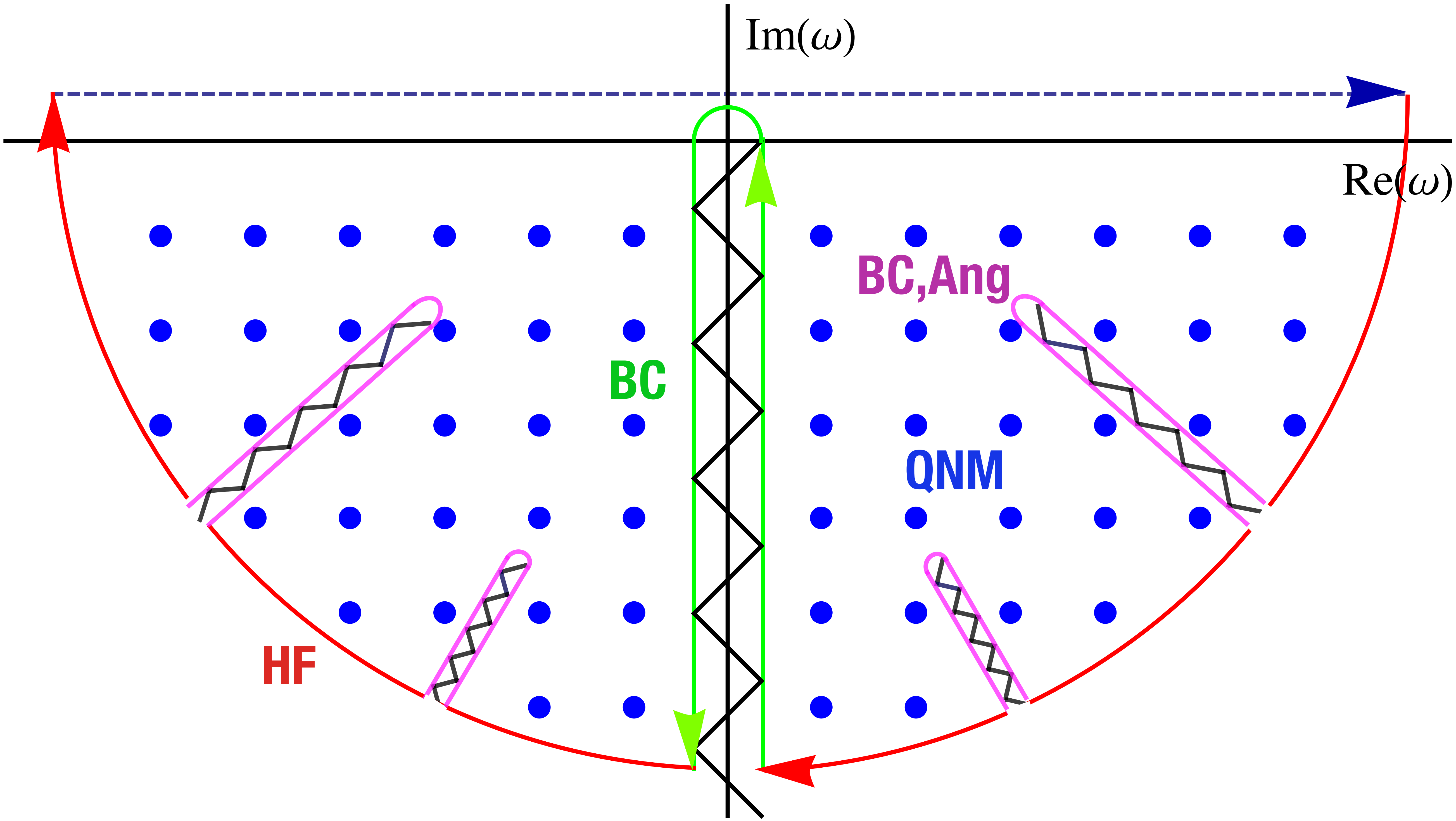
$$\omega_+ \cdot \omega < 0$$

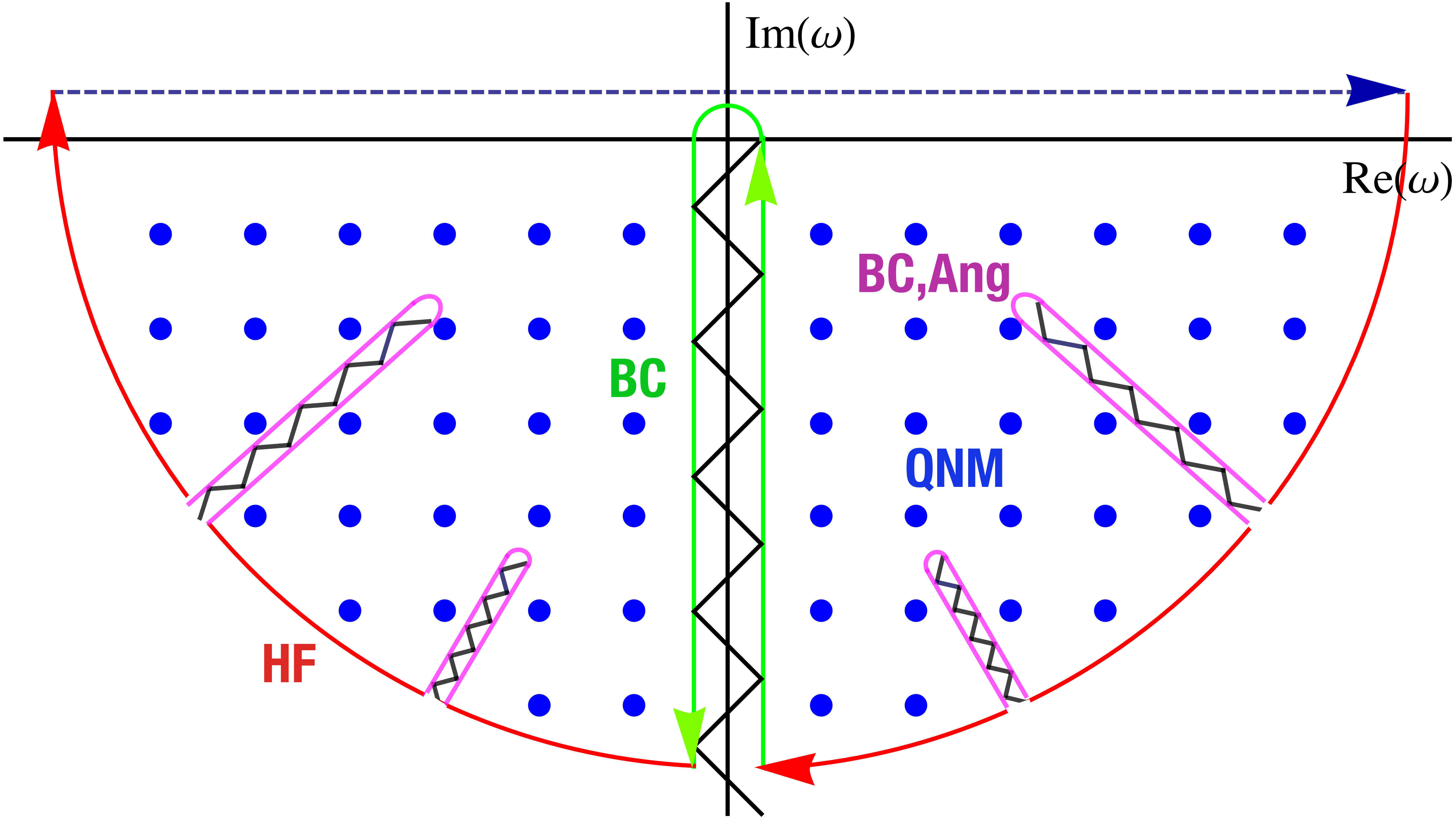
Field modes with those frequencies extract *rotational energy* from the BH thanks to existence of *ergosphere* (region near BH where ∂_t is spacelike)

Complex contour deformation

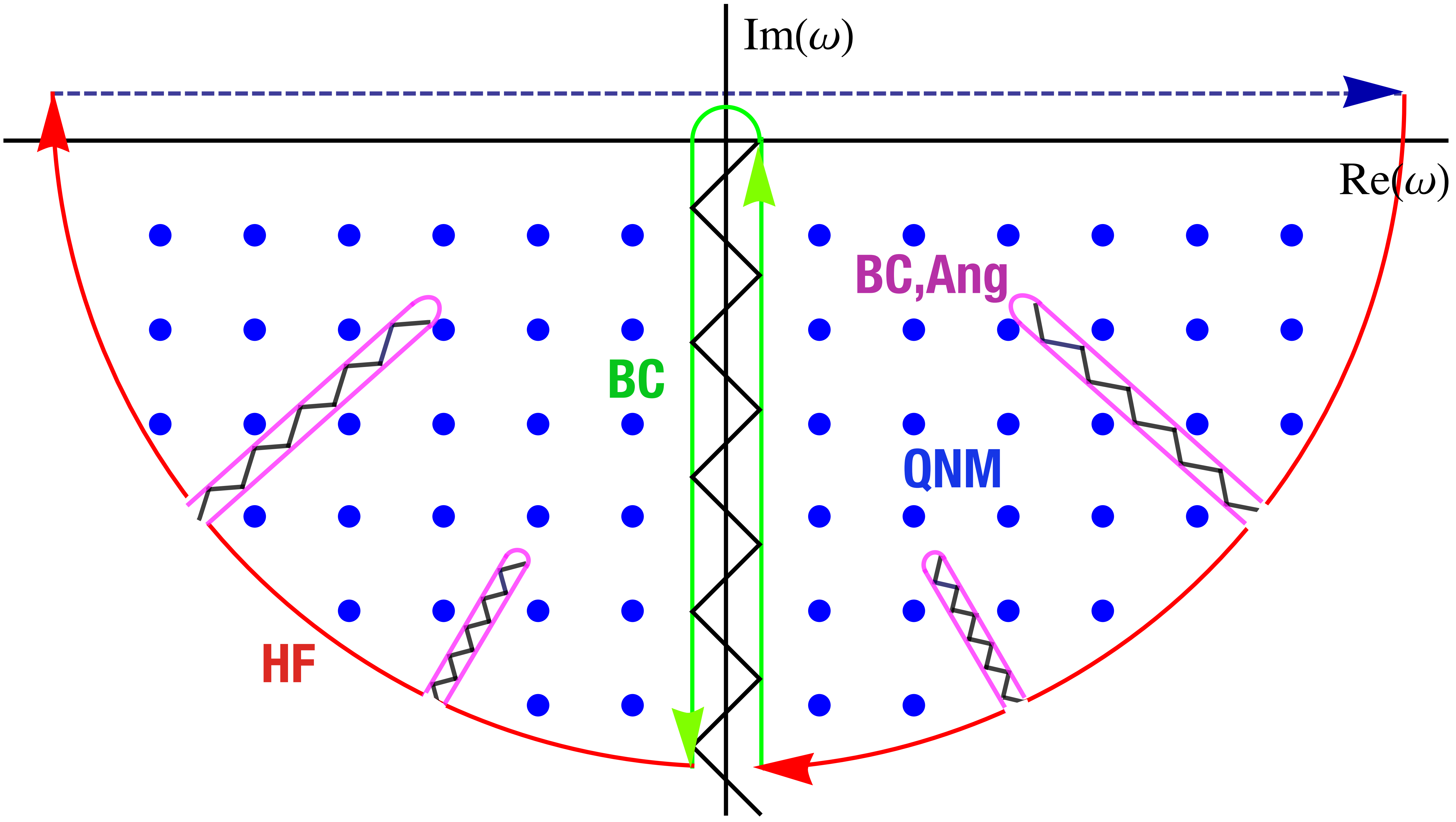
Instead of carrying out the Fourier integral along the real-ω axis, it's useful to deform the contour of integration into complex-ω plane

Then apply the residue th. to account for the singularities of the Fourier modes $G_{\ell m\omega}$


Im(ω)
Re(ω)
BC,Ang
BC
QNM
HF



$$G_{\ell m\omega} = G^{HF}_{\ell m\omega} + G^{QNM}_{\ell m\omega} + G^{BC}_{\ell m\omega} + G^{BC,Ang}_{\ell m\omega}$$



$$G_{\ell m\omega} = \xcancel{G^{HF}_{\ell m\omega}} + G^{QNM}_{\ell m\omega} + G^{BC}_{\ell m\omega} + \xcancel{G^{BC,Ang}_{\ell m\omega}}$$

Mode solutions

Mode slns. correspond to frequencies $\omega_{\ell mn} \in \mathbb{C}$ which are *poles* of the GF modes

$$G_{\ell m\omega}(r,r') = \frac{R^{in}_{\ell m\omega}(r_<)R^{up}_{\ell m\omega}(r_>)}{W} = \infty$$

$$\Big\uparrow$$

$$\omega = \omega_{\ell mn}$$

$$n = 0, 1, 2, \ldots$$

Mode solutions

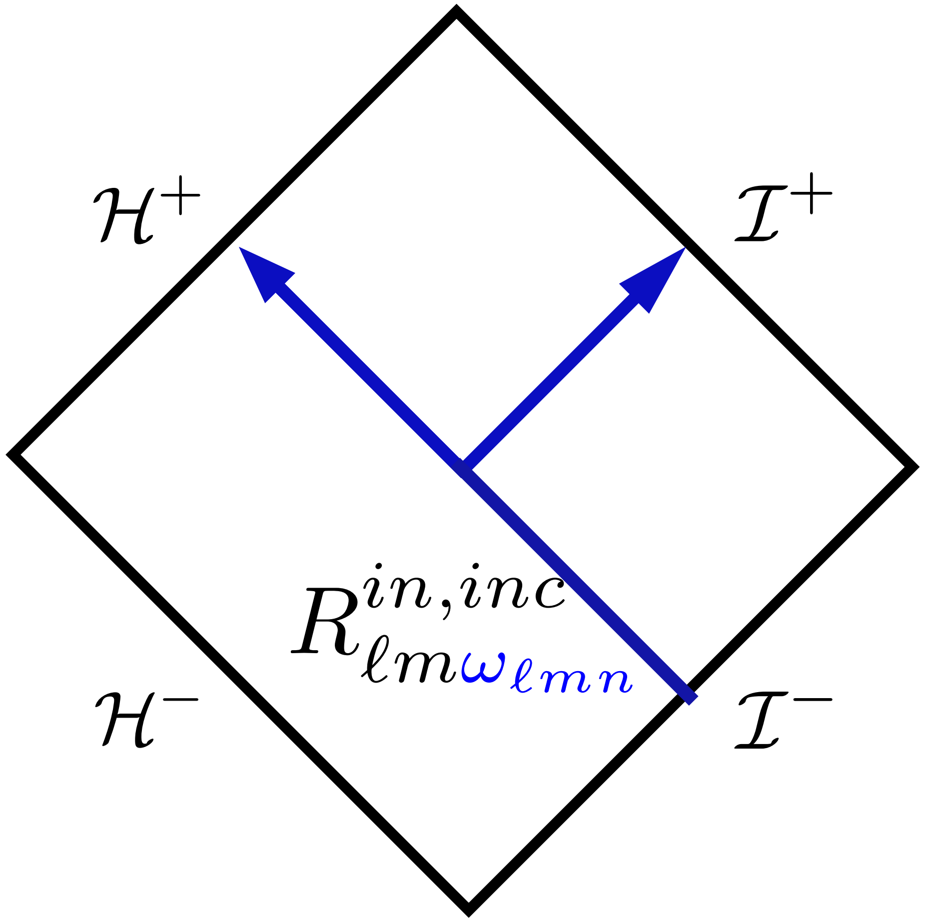
Mode slns. correspond to frequencies $\omega_{\ell mn} \in \mathbb{C}$ which are *poles* of the GF modes

So they are zeros of the denominator:

$$\begin{tikzcd}
G_{\ell m\omega}(r,r') = \dfrac{R^{in}_{\ell m\omega}(r_<)R^{up}_{\ell m\omega}(r_>)}{W} = \infty \\
W = 2i\omega R^{in,inc}_{\ell m\omega} = 0 \arrow[u, leftrightarrow, "\omega = \omega_{\ell mn}"']
\end{tikzcd}$$

$n = 0, 1, 2, \ldots$

Then: $e^{-i\omega_{\ell mn}t}R^{in}_{\ell m\omega_{\ell mn}}$



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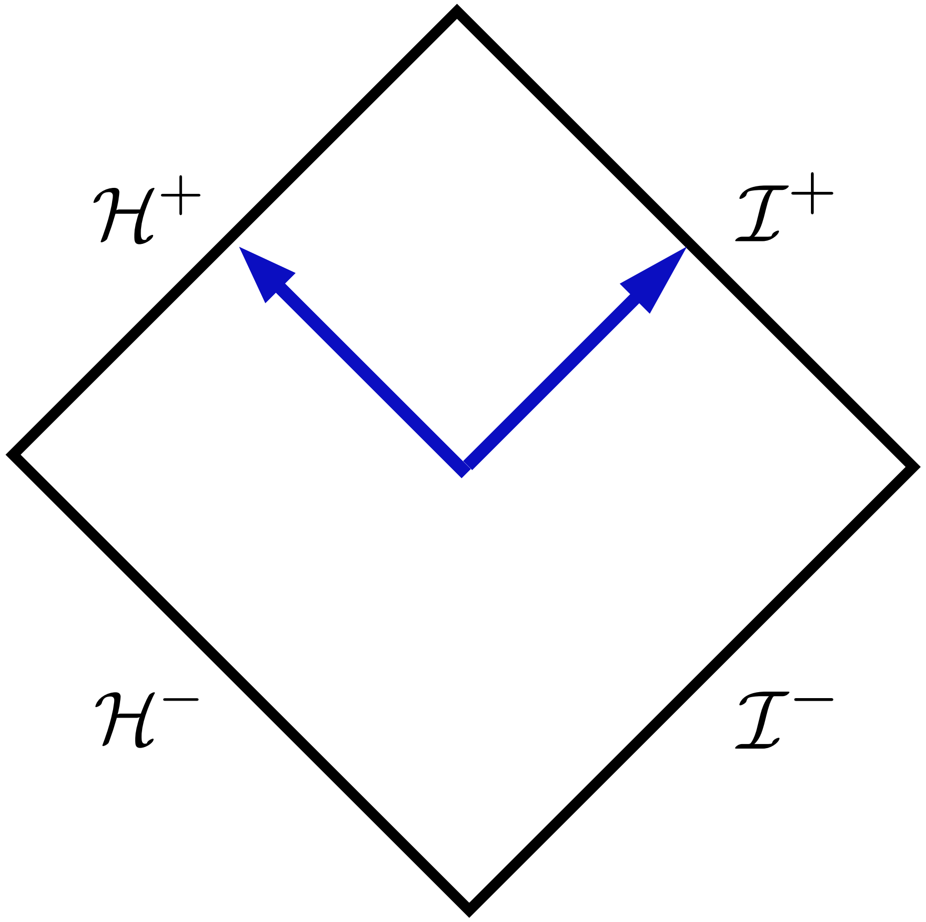

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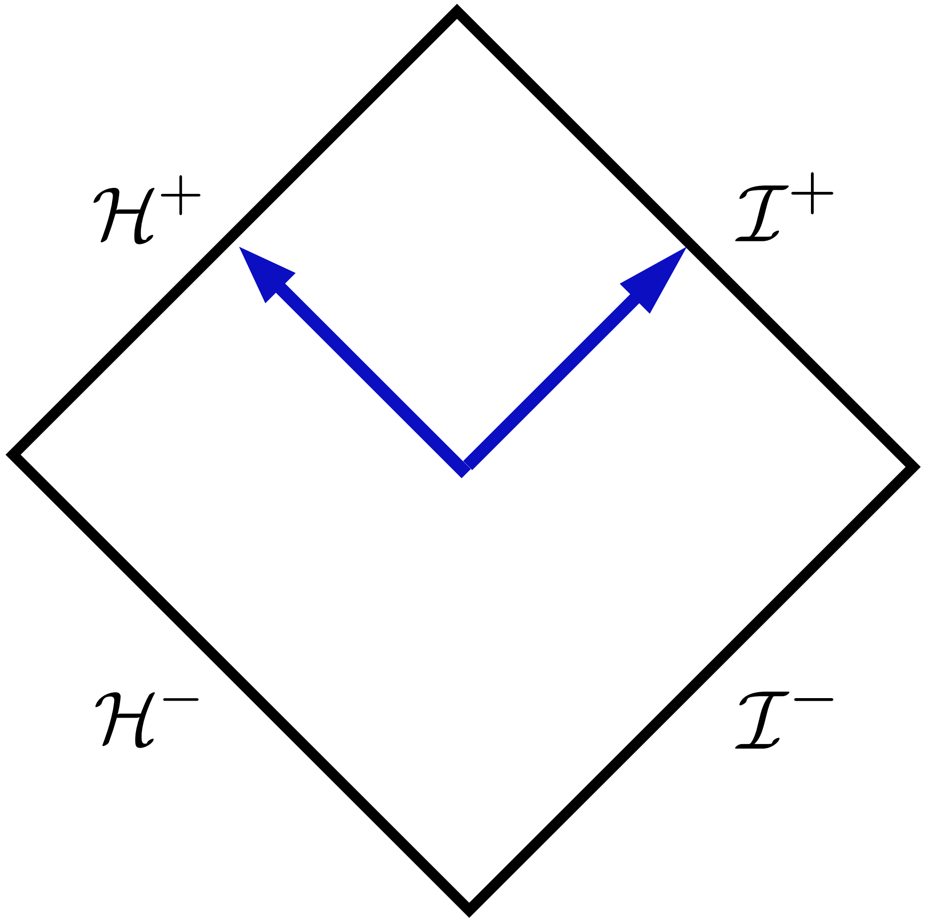
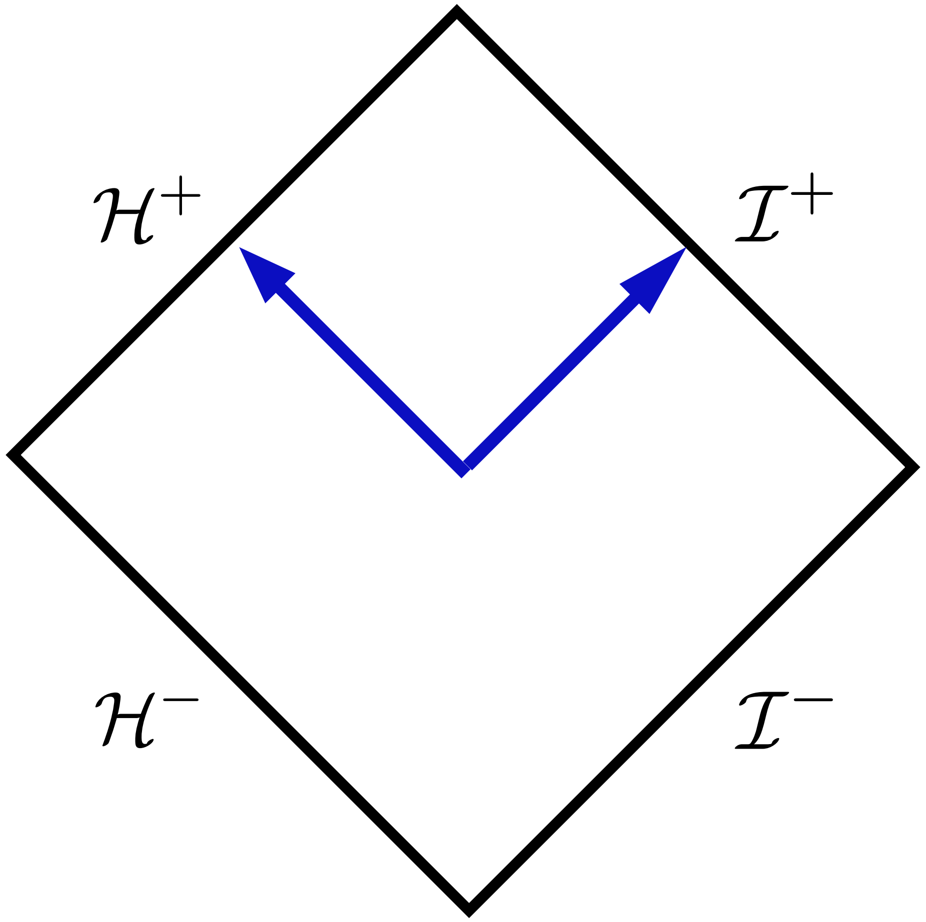


Mode slns. are purely ingoing waves into the horizon and purely ougoing at infinity

$$e^{-i\omega_+ r_*} \sim R^{in}_{\ell m\omega} \sim e^{i\omega r_*} \quad \text{at} \quad \omega = \omega_{\ell mn}$$

$$r_* \to -\infty \qquad r_* \to \infty$$

$$e^{-i\omega_{\ell mn}t}R^{in}_{\ell m\omega_{\ell mn}}$$



Time dependence:

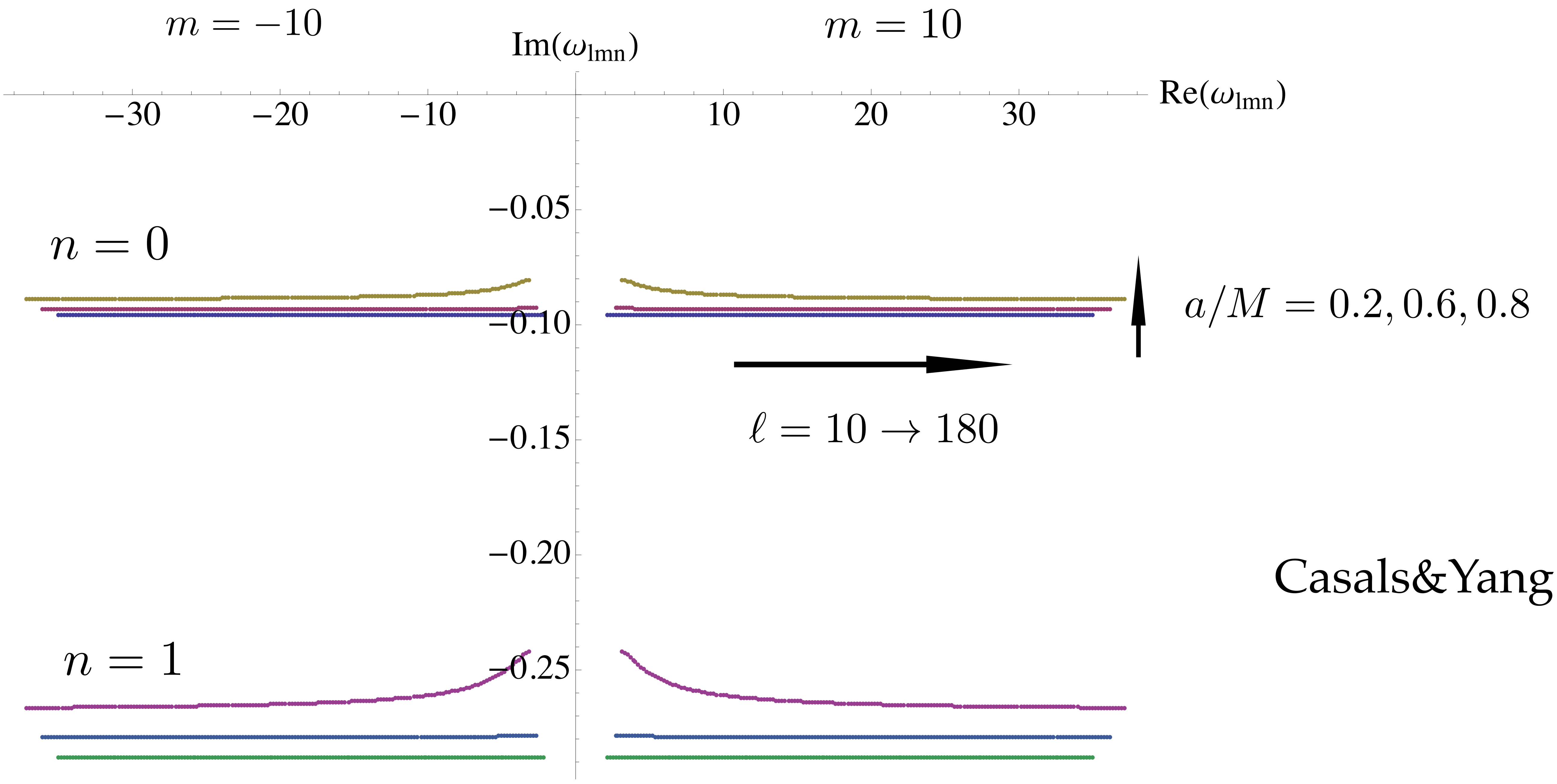
$$e^{-i\omega_{\ell mn}t}$$

$$\Big\downarrow \; t \rightarrow +\infty$$

$$\begin{cases} \text{If } Im(\omega_{\ell mn}) < 0 : \text{exponentially damped (quasinormal modes, QNMs)} \\ \text{If } Im(\omega_{\ell mn}) > 0 : \text{exponentially growing (unstable modes)} \end{cases}$$

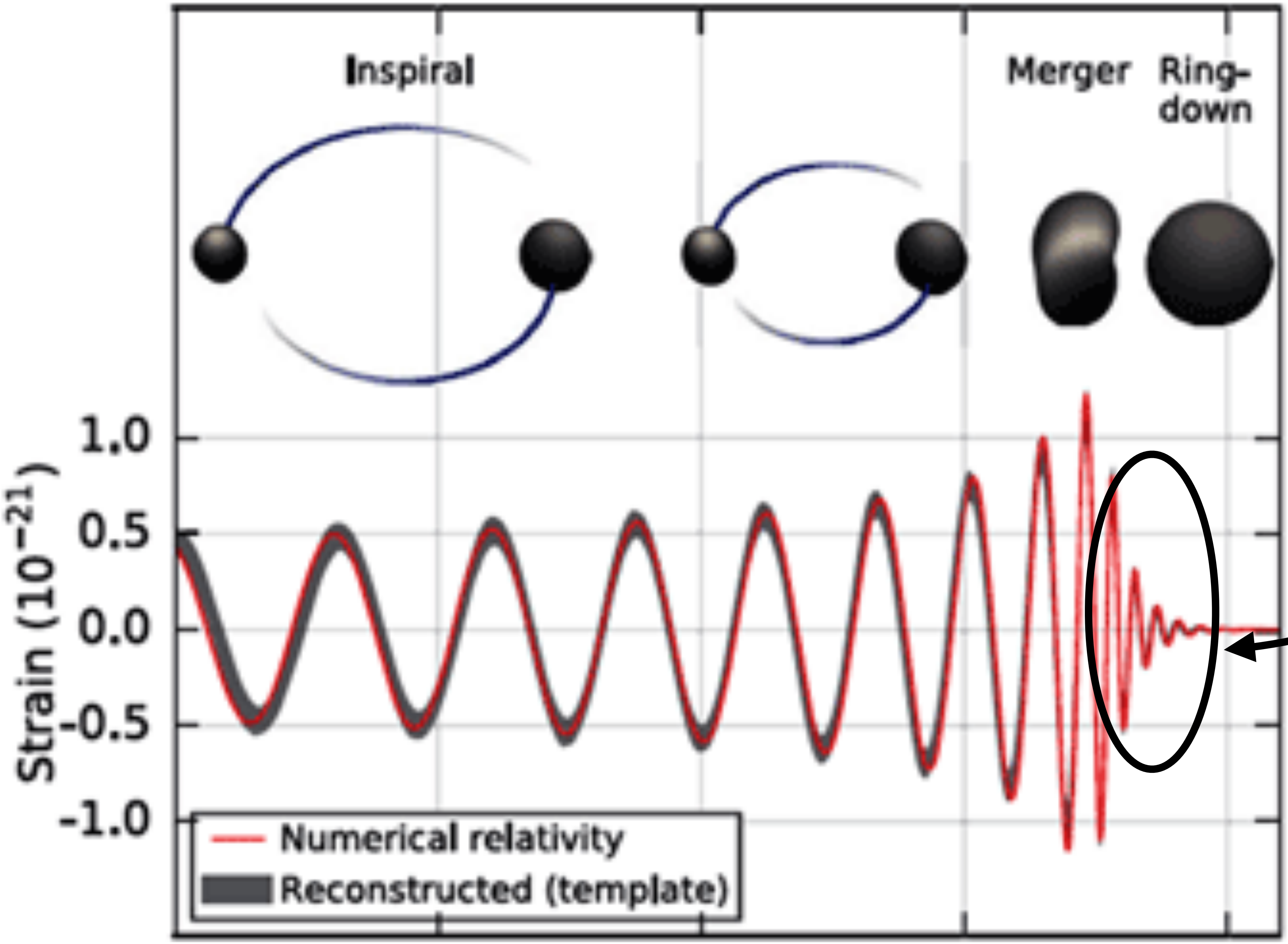
[If $Im\left(\omega_{\ell mn}\right) = 0$: *marginally* unstable]

QNMs $(Im(\omega_{\ell mn}) < 0$) of Kerr:



Casals&Yang

The last stage (*ringdown*) of a gravitational waveform can be modelled as perturbations of Kerr via QNMs:



perturbed Kerr emitting GWs

QNM measured by LIGO agrees with theoretical calculation

LIGO’16

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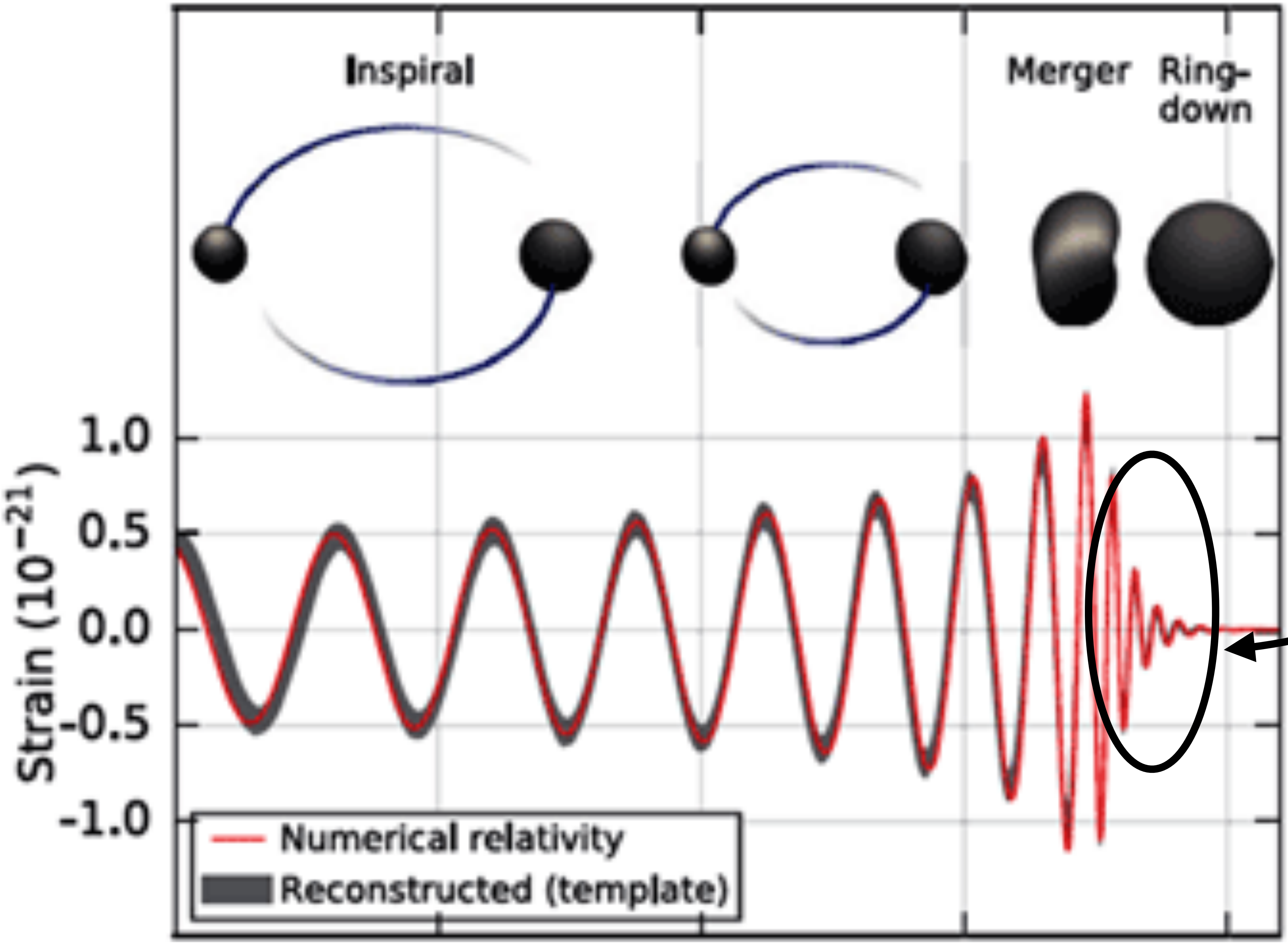

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Q: Are there any unstable modes ($Im(\omega_{\ell mn}) > 0$) in Kerr?

Some instabilities

- Kerr BH with event horizon removed by a “mirror” (may model wormholes) has unstable modes (Friedman’78)

Instability timescale ~ secs for massive wormholes (eg, Cardoso et al’08)

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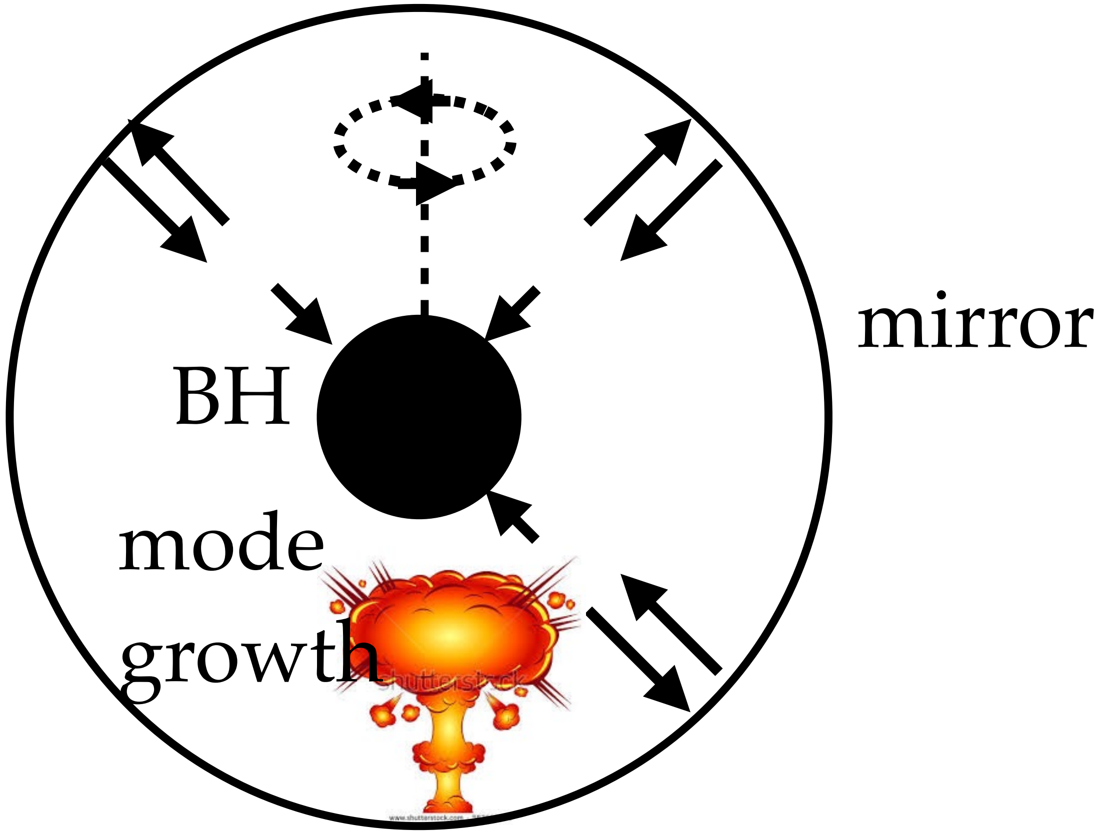
- BH 'bomb' (Press&Teukolsky'72):
Kerr BH unstable when surrounded by a (sufficiently far) "mirror"



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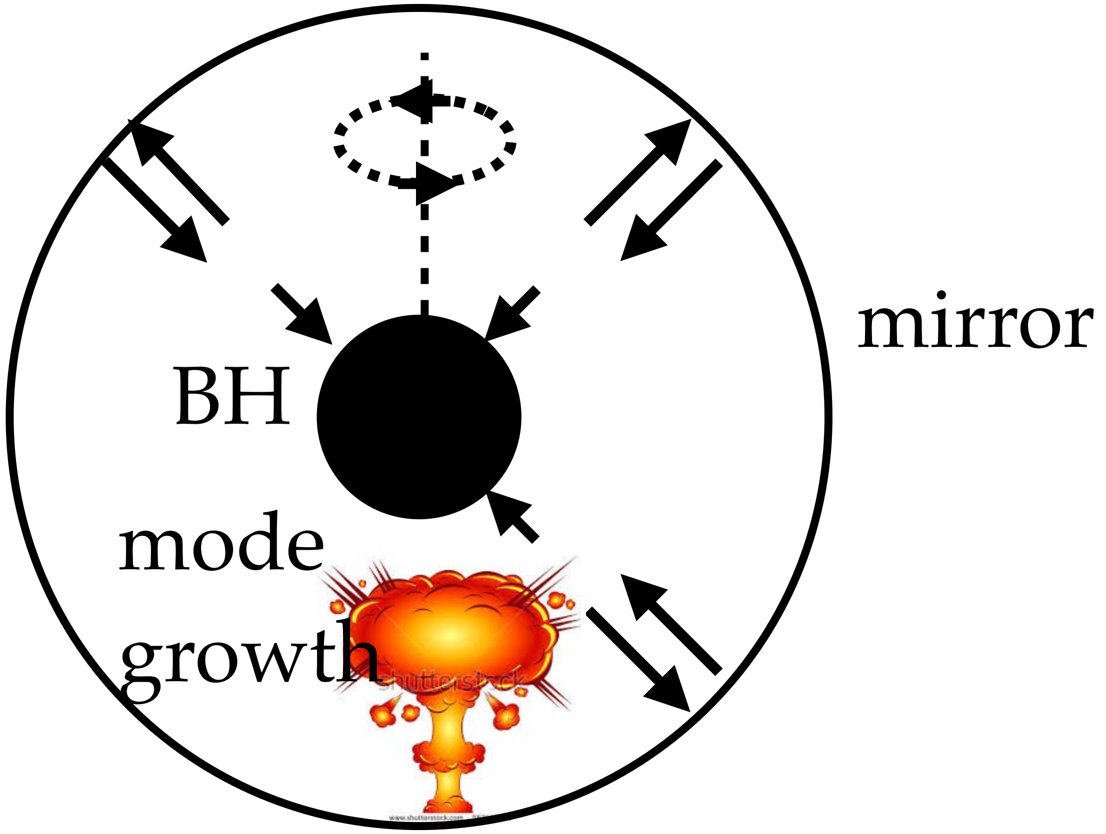

- Kerr BH is unstable under massive field perturbations (Damour et al'76)

Instability timescale has been used to constrain masses of fields, eg, mass of Proca field $\lesssim 4 \times 10^{-22}$ eV (Pani et al'12)

What about Kerr (*without* mirror) under *massless* field perturbations?

Mode stability of Kerr

Consider a possible unstable mode $\phi(x) = e^{im\varphi - i\omega t}\phi_{m\omega}(r,\theta)$

of the scalar wave eq. Multiply the eq. by ϕ^*, integrate over θ and r and take the Im part:

$$\int_{r_+}^{\infty} dr \int_{-1}^{+1} d(\cos\theta) \left[\frac{4amM}{r} - \frac{2\mathrm{Re}(\omega)}{r^2} \left[\left(r^2 + a^2\right)^2 - a^2 \Delta \sin^2\theta \right] \right] \frac{r^2}{\Delta} |\phi_{m\omega}|^2 = 0$$

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This equality requires: $0 < \mathrm{Re}(\omega) < |m|\Omega_+$

i.e., unstable modes are only possible in this (superradiant) regime

NB: for $a = 0 \rightarrow \Omega_+ = 0$: superradiant regime does not exist in Schwarzschild (it has no ergosphere) and so it's mode-stable

Whiting'89 came up with a smart way: an injective integral transformation

$$u(x) \equiv f_1(x) \int_{r_+}^{\infty} f_2(r,x) e^{i\omega r} R(r) dr$$

simple specific functions

satisfies radial Teukolsky ODE
(has superradiance) $\hat{\mathcal{O}}_r \; R(r) = 0$

satisfies a new radial ODE that has *no* superradiance (it corresponds to a spacetime *without* ergosphere)

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This allows to show it must be $u \equiv 0$ if R is an unstable mode

$$\Big\downarrow \text{injectivity}$$

$$R \equiv 0$$

So there're no unstable modes (nor marginally unstable, Shlapentokh-Rothman'15) for massless general-spin fields in Kerr =>Kerr is mode-stable

Analogy (?) Kerr - SQCD

Aminov, Grassi & Hatsuda'20 noticed that the radial Teukolsky operator can be rewritten as

$$\hat{\mathcal{O}}_r = z(z-1)\frac{d^2}{dz^2} - p^2 z(z-1) - m_3\, p\,(2z-1) + \left(E + \frac{1}{4}\right) -$$

$$\frac{m_1 m_2}{z} - \frac{(m_1+m_2)^2 - 1}{4z(z-1)}$$

where $z \equiv \dfrac{r - r_-}{r_+ - r_-}$

$$r = (r_-, r_+, \infty) \leftrightarrow z = (0, 1, \infty)$$

$$E \equiv -\lambda - a^2\omega^2 - s^2 + 8M^2\omega^2 - \frac{1}{4}$$

$$p \equiv i\omega(r_+ - r_-)$$

and the 'mass' parameters $m_1 = s - \xi_- + \xi_+$ $m_2 = \xi_- + \xi_+$

$$\xi_\pm \equiv i\,\frac{\omega_\pm}{2\,\kappa_\pm}$$

$$m_3 = -s - \xi_- + \xi_+$$

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$$\xi_\pm \equiv i\,\frac{\omega_\pm}{2\,\kappa_\pm} \qquad m_3 = -s - \xi_- + \xi_+$$

The ODE is symmetric under $m_1 \leftrightarrow m_2$ (but *not* under $m_{1,2} \leftrightarrow m_3$)

This ODE is the same as the quantum SU(2) Seiberg–Witten eq. with 3 fundamental hypermultiplets in supersymmetric quantum chromodynamics (SQCD), where the m_i are "flavour masses" of the hypermultiplets

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They *conjectured* that mode frequencies of radial Teukolsky eq. in Kerr are *equivalent* to a Bohr-Sommerfeld-like rule for resonances in the SQCD

Since this rule in SQCD is manifestly symmetric under *any* $m_i \leftrightarrow m_j$ $\forall i, j \in \{1, 2, 3\}$

they *conjectured* that the mode sln. condition in Kerr is also symmetric under any exchange of SQCD masses

This ODE is the same as the quantum SU(2) Seiberg–Witten eq. with 3 fundamental hypermultiplets in supersymmetric quantum chromodynamics (SQCD), where the m_i are "flavour masses" of the hypermultiplets

They *conjectured* that mode frequencies of radial Teukolsky eq. in Kerr are *equivalent* to a Bohr-Sommerfeld-like rule for resonances in the SQCD

Since this rule in SQCD is manifestly symmetric under *any* $m_i \leftrightarrow m_j$ $\forall i, j \in \{1, 2, 3\}$

they *conjectured* that the mode sln. condition in Kerr is also symmetric under any exchange of SQCD masses

Casals & Teixeira da Costa'22 *proved*, directly in Kerr, this SQCD-mass symmetry conjecture and used it to obtain an alternative proof of mode stability as follows…

Kerr: "SQCD-mass" symmetries & mode stability

First rewrite the b.c. for mode slns. in terms of the new 'SQCD variables':

$$e^{-\frac{1}{2}(m_1+m_2-1)\ln(z-1)} \sim R^{in}_{\ell m\omega} \sim e^{pz}$$

$$\begin{array}{cc} r_* \to -\infty & r_* \to \infty \\ (z \to 1) & (z \to \infty) \end{array}$$

So the b.c. for mode slns., like the ODE, are symmetric under $m_1 \leftrightarrow m_2$ but *not* under $m_{1,2} \leftrightarrow m_3$

Next express the ‘in’-sln. as a (Jaffé) expansion about $r = r_+$ (Leaver´85):

$$R^{in}_{\ell m\omega} = e^{p(z-1)} e^{\frac{1}{2}(m_1+m_2+2m_3-1)\ln z} e^{-\frac{1}{2}(m_1+m_2-1)\ln(z-1)} \sum_{n=0}^{\infty} b_n \left(\frac{z-1}{z}\right)^n$$

with coeffs. satisfying a 3-term recurrence rln.

$$\alpha_n^{(+1)} b_{n+1} + \alpha_n^{(0)} b_n + \alpha_n^{(-1)} b_{n-1} = 0$$

The coefficients $\alpha_n^{(\pm 1)}$ and $\alpha_n^{(0)}$ depend on M, a, ω, ℓ, m

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In order to also satisfy the mode sln. b.c. also at $r = \infty$ $(z = \infty)$

we need for the infinite n-series to converge there

It can be shown that requiring for the series to converge at $r = \infty$ is equivalent to the mode freqs. $\omega = \omega_{\ell m n}$ satisfying a continued fraction eq.

$$0 = \alpha_0^{(0)} + \cfrac{-\alpha_0^{(+1)} \alpha_1^{(-1)}}{\alpha_1^{(0)} + \cfrac{-\alpha_1^{(+1)} \alpha_2^{(-1)}}{\alpha_2^{(0)} + \cfrac{-\alpha_2^{(+1)} \alpha_3^{(-1)}}{\cdots}}}$$

where

$$\alpha_{n-1}^{(+1)} \alpha_n^{(-1)} = n \left\{ [n - \sigma_1][n(n - \sigma_1) + \sigma_2] + \sigma_3 \right\}$$

$$\alpha_n^{(0)} = E + \frac{1}{4} + 2n(p - n) + (2n + 1 - p)\sigma_1 - \sigma_2$$

$$\sigma_1 \equiv m_1 + m_2 + m_3 \qquad \sigma_2 \equiv m_1 m_2 + m_1 m_3 + m_2 m_3$$

$$\sigma_3 \equiv m_1 m_2 m_3$$

are invariant under *any* exchange $m_i \leftrightarrow m_j$, with $i, j \in \{1, 2, 3\}$

Therefore, the mode freqs. $\omega_{\ell m n}$ are also symmetric under $m_i \leftrightarrow m_j$

Remember the SQCD masses:

$$m_1 = s - \xi_- + \xi_+ \qquad m_2 = \xi_- + \xi_+ \qquad m_3 = -s - \xi_- + \xi_+$$

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This thus provides a new, simpler (no need for integral nor differential transformations!) proof of mode stability of (subextremal) Kerr BHs

Stability of Extremal Kerr

All results so far were for *sub*extremal Kerr. In *extremal* Kerr $(a = M)$:

- There're no unstable modes in extremal Kerr [Teixeira da Costa'20 by extending Whiting's integral transformation]

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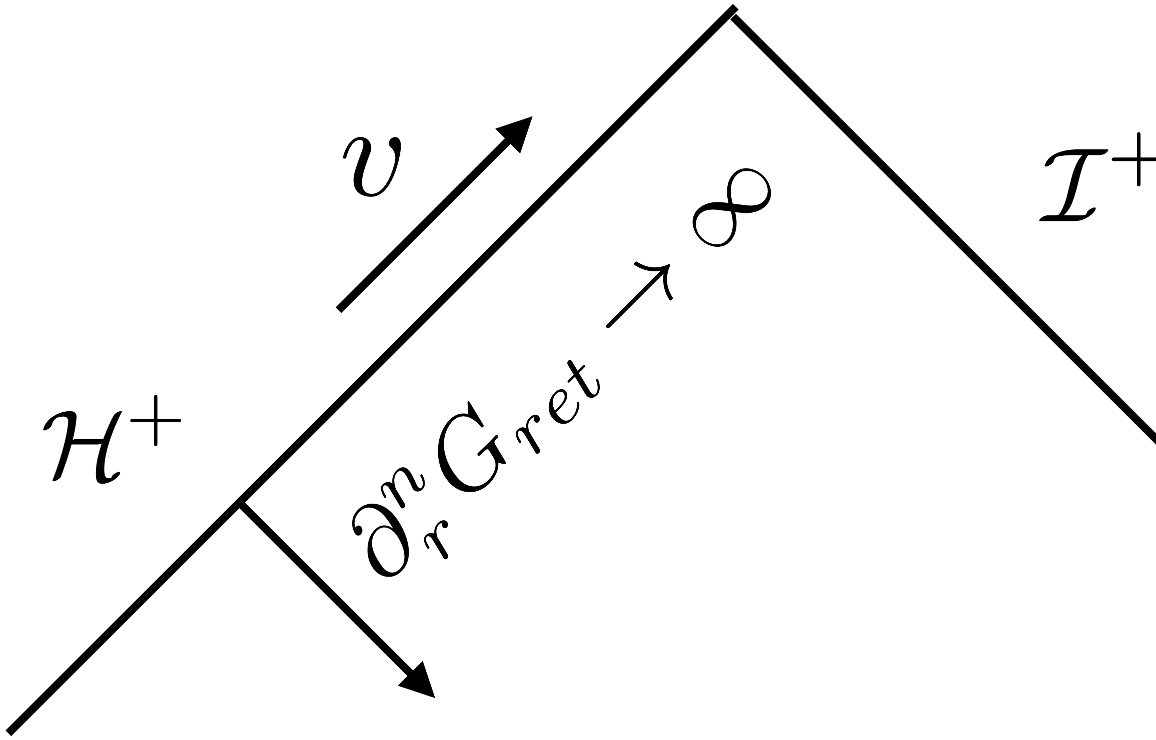
$$\left(\partial_r^n G_{ret}\right)\Big|_{\mathcal{H}} \sim v^{n-s-1/2} \quad \text{as} \quad v \to \infty$$



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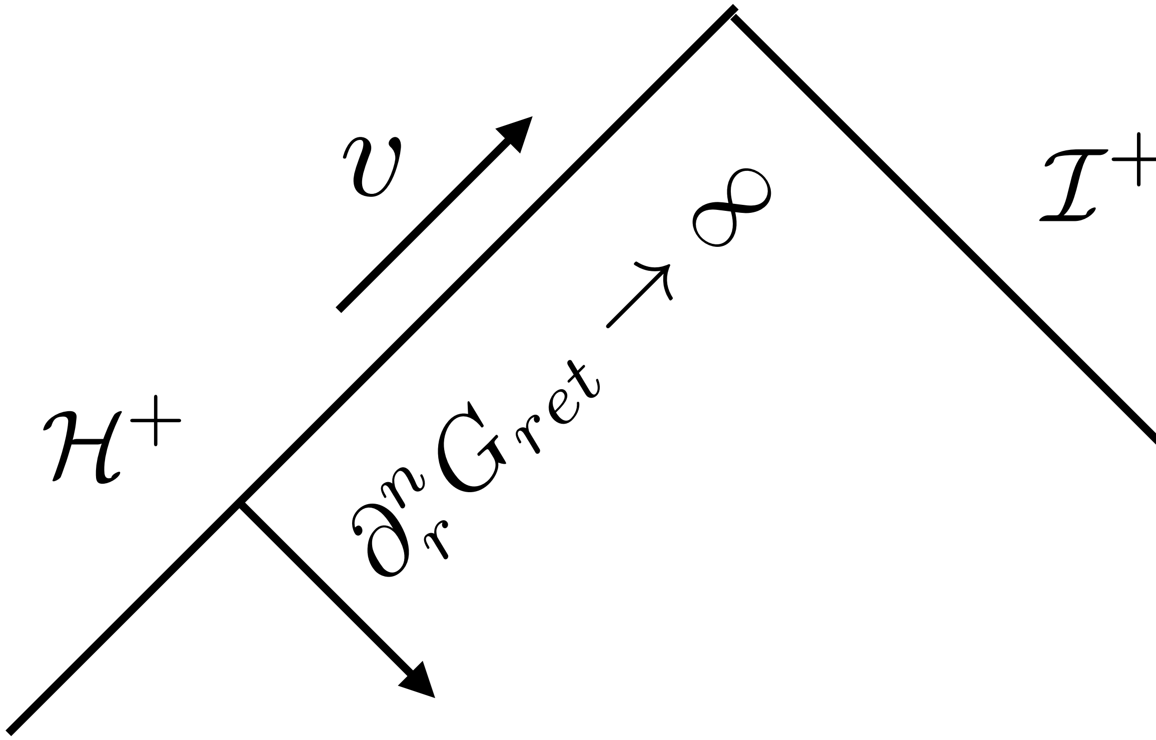

- Such behaviour is due to an emergent near-horizon conformal symmetry (AdS_2) of extremal BHs [Gralla & Zimmerman'18]

Full linear stability of Kerr?

Note: the full linear sln. of Teukolsky eq. is obtained from *infinite* sums/integrals of frequency modes $\sum_{\ell,m}\int_{-\infty}^{\infty}d\omega$ so non-existence of unstable modes does not guarantee *full linear stability*

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Kerr is fully (ie, not just modal) linearly stable under scalar perturbations (Dafermos, Rodnianski & Shlapentokh-Rothman'16)

Open question: full linear stability of Kerr under gravitational perturbations

1. BH Perturbations

2. Stability properties of Kerr

3. Stability properties of Kerr-de Sitter

4. Conclusions

Kerr-de Sitter Black Holes

Kerr-de Sitter metric represents a rotating BH in a de Sitter Universe

$$(M, a, \underset{\substack{\uparrow \\ \text{Cosmological const.}}}{\Lambda})$$

It's similar to the Kerr metric but with 3(+1) horizons:

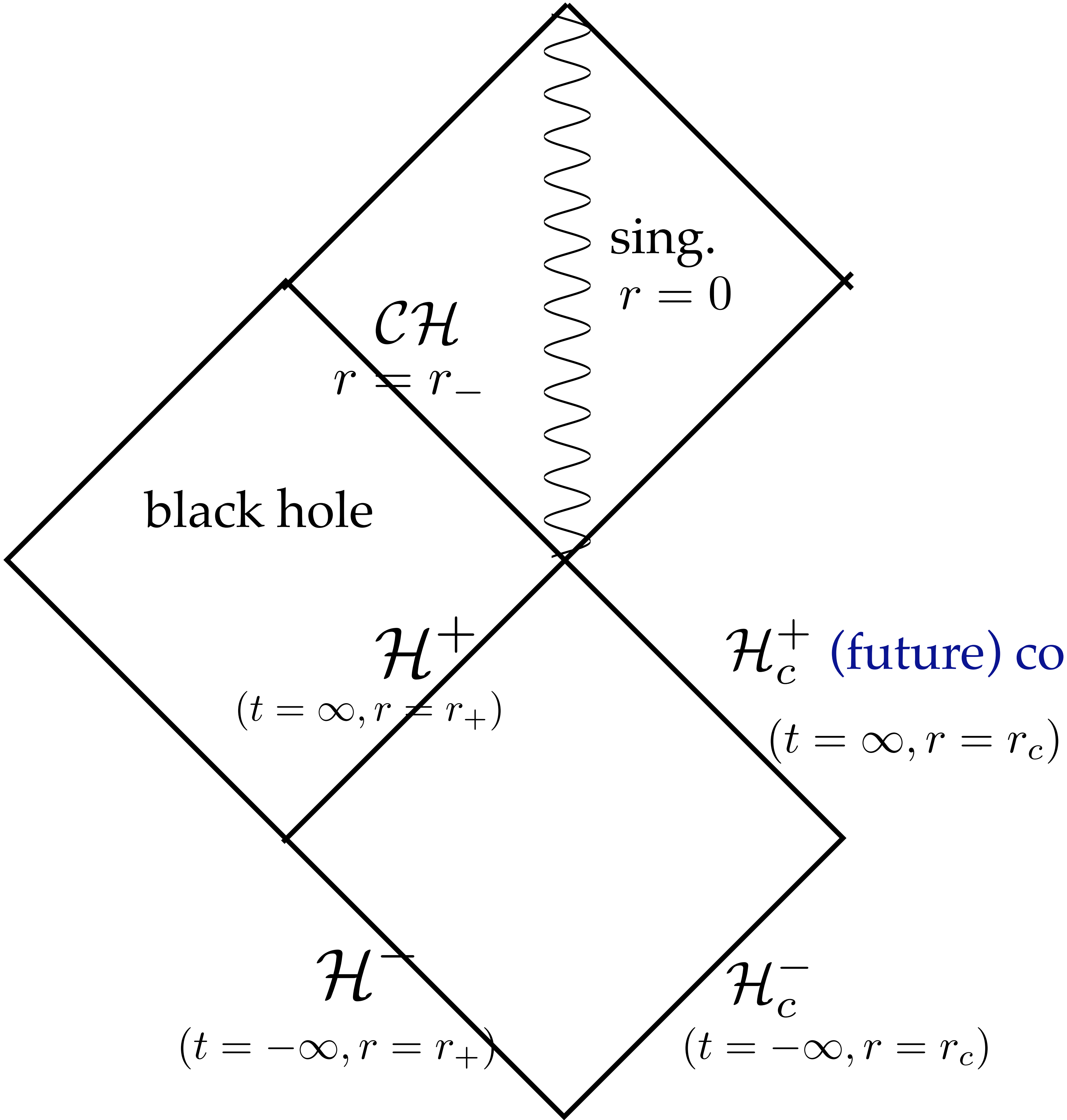
$$\Delta = (r^2 + a^2)\left(1 - \frac{\Lambda r^2}{3}\right) - 2Mr = -\frac{\Lambda}{3}(r - r_{--})(r - \underset{\substack{\uparrow \\ \text{Cauchy} \\ \text{horizon}}}{r_-})(r - \overset{\substack{\text{event horizon} \\ \downarrow}}{r_+})(r - \underset{\substack{\uparrow \\ \text{cosmological} \\ \text{horizon}}}{r_c})$$

$$r_{--} < 0 < r_- < r_+ < r_c$$

Each horizon j= - -, - , +, c has

angular velocity $\Omega_j = \dfrac{a}{r_j^2 + a^2}$ and surface gravity κ_j

Carter-Penrose diagram of Kerr-de Sitter



Perturbations of Kerr-de Sitter

Massless spin-field perturbations ψ of Kerr-de Sitter obey a Teukolsky master PDE (similar to the one in Kerr)

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Again, it separates by variables

$$G_{ret} = \sum_{\ell,m} \int_{-\infty}^{\infty} d\omega\ e^{im\varphi - i\omega t} S_{\ell m\omega}(\theta) S_{\ell m\omega}(\theta') G_{\ell m\omega}(r, r')$$

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Casals & Teixeira da Costa'22 analyzed it, found mass symmetries and used them to investigate the mode stability of Kerr-de Sitter as follows

`Mass' symmetries in Kerr-de Sitter

We expressed the radial Teukolsky operator in Kerr-de Sitter as

$$\hat{\mathcal{O}}_r = z(z-1)(z-z_2)\frac{d^2}{dz^2} + \frac{(4E-1)z_2+1}{4} - \frac{1}{(z-z_2)}\left[\frac{1}{4}z(z-1)((m_3+m_4)^2-1) - m_3 m_4 (z-z_2)\left(z-\frac{1}{2}\right)\right] - \frac{1}{z(z-1)}\left[m_1 m_2\left(\frac{z}{2}-z_2\right)(z-1) + \frac{1}{4}(z-z_2)((m_1+m_2)^2-1)\right]$$

where $$z \equiv z_\infty \frac{r-r_-}{r-r_{--}} \qquad z_2 \equiv z_\infty \frac{r_c-r_-}{r_c-r_{--}} \qquad z_\infty \equiv \frac{r_+-r_{--}}{r_+-r_-}$$

$$r = (r_{--}, r_-, r_+, r_c, \infty) \leftrightarrow z = (\infty, 0, 1, z_2, \infty) \qquad E = E\left(\lambda, \{\xi_j\}, \{r_j\}\right)$$

and we now have *four* 'mass' parameters:

$$m_1 = s - \xi_- + \xi_+ \qquad m_3 = -s - \xi_- + \xi_+ \qquad \xi_j \equiv i\,\frac{\omega_j}{2\,\kappa_j}$$

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The ODE is symmetric under $m_1 \leftrightarrow m_2$ or $m_3 \leftrightarrow m_4$ (but *not* others)

The radial Teukolsky ODE in KdS in principle has 5 regular singular pts. but it can be transformed into a Heun eq. with just 4 *regular* singular pts.:

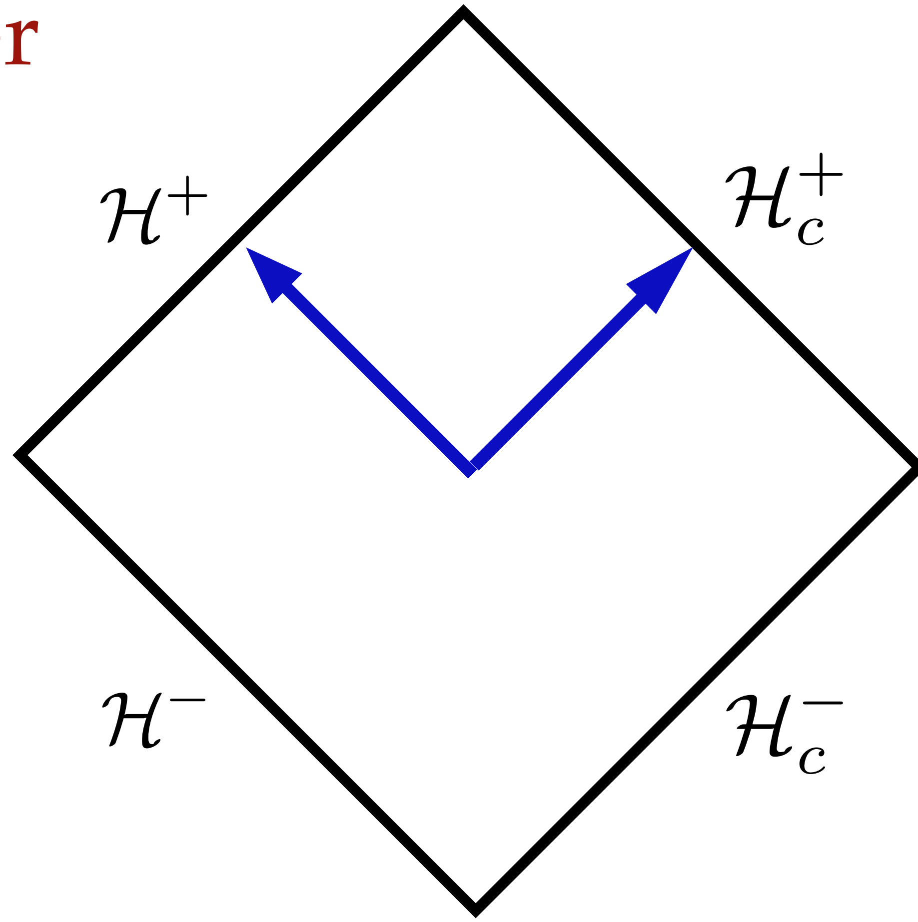
$$r = (r_{--}, r_{-}, r_{+}, r_{c})$$

($r = \infty$ is a *removable* singularity)

it's not a confluent Heun eq. as the irregular sing. $r = \infty$ in Kerr has split into 2 regular sing. pts. here

Mode slns. in Kerr-de Sitter

B.c. for mode slns. ($\omega = \omega_{\ell m n}$): purely ingoing waves into the event horizon and purely outgoing at the cosmological horizon



$$e^{-i\omega_+ r_*} \sim R^{in}_{\ell m\omega} \sim e^{i\omega_c r_*}$$

$$r_* \to -\infty \qquad r_* \to \infty$$

$$(r \to r_+) \qquad (r \to r_c)$$

$$\frac{dr_*}{dr} \equiv \left(1 + \frac{a^2\Lambda}{3}\right) \frac{(r^2 + a^2)}{\Delta}$$

Re-expressed in the new variables:

$$e^{-\frac{1}{2}(m_1+m_2-1)\ln(z-1)} \sim R^{in}_{\ell m\omega} \sim e^{-\frac{1}{2}(m_3+m_4-1)\ln(z-z_2)}$$

$$z \to 1 \qquad z \to z_2$$

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So the b.c. for mode slns., like the ODE, are symmetric under $m_1 \leftrightarrow m_2$ or $m_3 \leftrightarrow m_4$ (but *not* others)

Instead of the Jaffé expansion about $r = r_+$ which we used in Kerr for

$R^{in}_{\ell m\omega}$ we use an expansion in terms of *confluent hypergeometric funcs.*

$$R^{in}_{\ell m\omega} \propto \sum_{n=0}^{\infty} b_n \, {}_2F_1 \left(-n, n+1-\textstyle\sum_{j=1}^{4} m_j ; 1-m_1-m_2 ; \frac{z-1}{z_2-1} \right)$$

where the coefficients satisfy a 3-term recurrence rln.

$$\alpha_n^{(+1)} b_{n+1} + \alpha_n^{(0)} b_n + \alpha_n^{(-1)} b_{n-1} = 0$$

and $\alpha_n^{(\pm 1)}$ and $\alpha_n^{(0)}$, like the ODE, are invariant under $m_1 \leftrightarrow m_2$ or $m_3 \leftrightarrow m_4$ (but *not* others)

Similarly to Kerr, the above expression corresponding to a mode sln. is equivalent to the convergence of the series in $r \in [r_+, r_c]$

Convergence of the series corresponds to a continued fraction eq. :

$$0 = \alpha_0^{(0)} + \cfrac{-\alpha_0^{(+1)}\alpha_1^{(-1)}}{\alpha_1^{(0)} + \cfrac{-\alpha_1^{(+1)}\alpha_2^{(-1)}}{\alpha_2^{(0)} + \cfrac{-\alpha_2^{(+1)}\alpha_3^{(-1)}}{\cdots}}}$$

where

$$\alpha_{n-1}^{(+1)}\alpha_n^{(-1)} = \frac{n^2(n-\sigma_1)^2\left[\sigma_1\sigma_3 - 4\sigma_4 + (\sigma_2 + (n-\sigma_1)n)^2\right]}{(\sigma_1-2n)^2[(\sigma_1-2n)^2-1]} + \frac{n(n-\sigma_1)\left[\sigma_1\left(\sigma_2\sigma_3 - \sigma_1\sigma_4\right) - \sigma_3^2\right]}{(\sigma_1-2n)^2[(\sigma_1-2n)^2-1]}$$

$$\alpha_n^{(0)} = \frac{[\sigma_1^3 - 4\sigma_1\sigma_2 + 8\sigma_3]\sigma_1}{8(-2+\sigma_1-2n)(\sigma_1-2n)} + \frac{2(4E+1)z_2 + 2 + (z_2+1)(\sigma_1-2n-2)(\sigma_1-2n) - 2z_2[\sigma_1^2 - 2\sigma_2]}{8(z_2-1)}$$

$$\sigma_1 \equiv \sum_{j=1}^{4} m_j \qquad \sigma_2 \equiv \prod_{i,j=1,j\neq i}^{4} m_i m_j \qquad \sigma_3 \equiv \prod_{i,j,k=1,k\neq j\neq i}^{4} m_i m_j m_k$$

$$\sigma_4 \equiv \prod_{j=1}^{4} m_j$$

are symmetric under *any* exchange $m_i \leftrightarrow m_j$, with $i, j \in \{1, 2, 3, 4\}$

Therefore, the mode freqs. $\omega_{\ell mn}$ are also symmetric under $m_i \leftrightarrow m_j$

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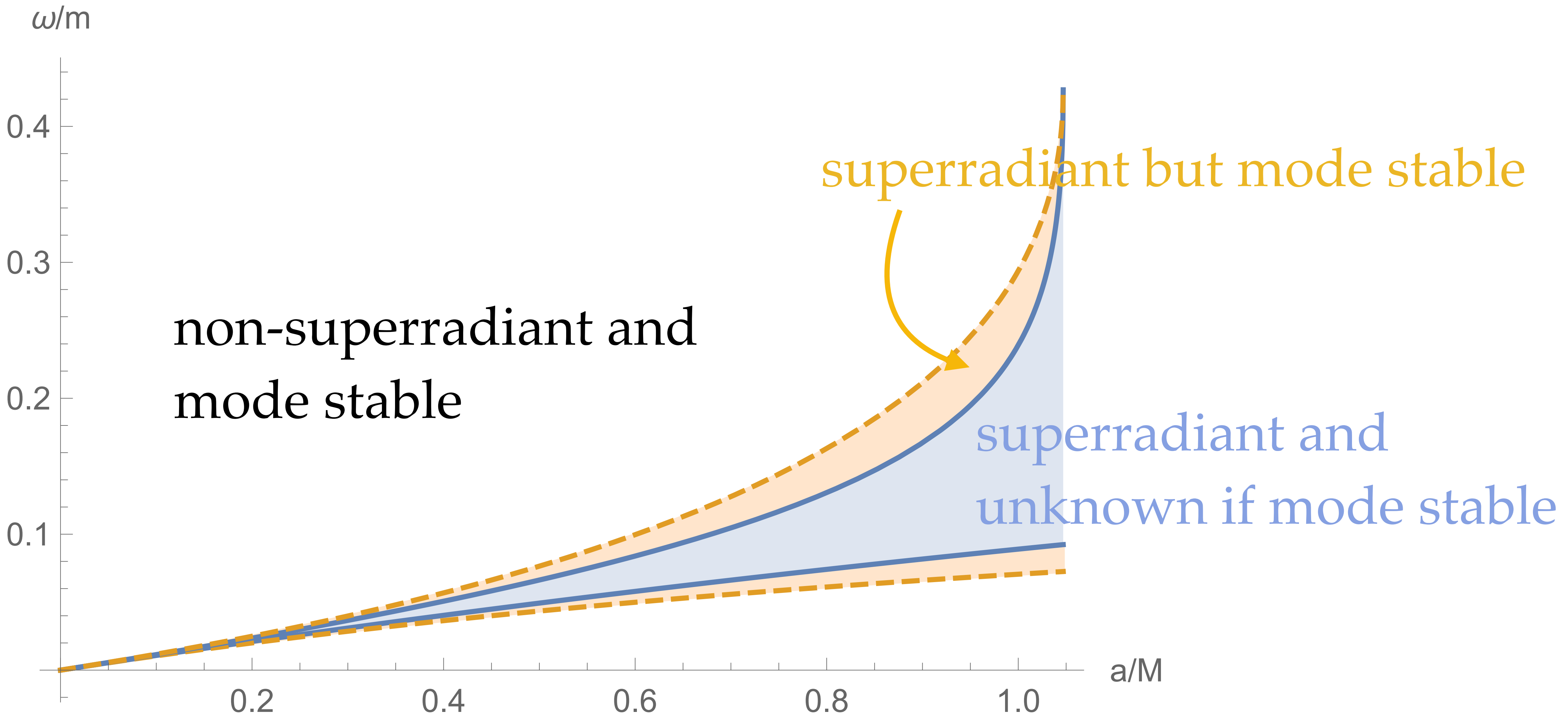
Using $m_2 \leftrightarrow m_3$ we transformed to a radial ODE which allowed us to exclude unstable modes from a subregion of the superradiant frequency regime



superradiant but mode stable

non-superradiant and mode stable

superradiant and unknown if mode stable

Open question: exclude unstable modes from the blue region so as to prove the mode stability of Kerr-de Sitter

1. BH Perturbations

2. Stability properties of Kerr

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4. Conclusions

Conclusions

- Kerr: mode stability proven

  Open question: prove *full* linear stability under grav. perturbations

- Kerr-de Sitter: only *partial* mode stability proven

  Open question: *complete* proof of mode stability

Apart from the explicit (Killing) symmetries, Kerr harbours hidden symmetries (Killing Yano, 'SQCD mass', near-horizon conformal geometry, etc) which regularly surprise us and allow us to make analytical progress on the understanding of astrophysical BHs

Merci bien!