A Few Numerical Aspects for the Simulation of Superradiance and Hyperradiance in Charged Symmetric Black Holes

Laurent Di Menza

Laboratoire de Mathématiques de Reims

Joint work with Jean-Philippe Nicolas

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Plan of the talk

Motivation

Let us consider the Cauchy problem for the one-dimensional wave equation including a spatial potential V :

$$\begin{cases} \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} + V(x)u = 0 & x \in \mathbb{R} \quad t > 0 \\ u(0,x) = u_0(x) & x \in \mathbb{R} \\ \dfrac{\partial u}{\partial t}(0,x) = u_1(x) & x \in \mathbb{R} \end{cases}$$

with $c > 0$ and where u_0 and u_1 are given functions of x.

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with $c > 0$ and where u_0 and u_1 are given functions of x.

Conservative equation

$$E(t) = \frac{1}{2}\int_{\mathbb{R}} \left(\left(\frac{\partial u}{\partial t}(t,x)\right)^2 + c^2 \left\{ \left(\frac{\partial u}{\partial x}(t,x)\right)^2 + V(x)u(t,x)^2 \right\} \right) dx$$

Motivation

Assumptions

$V \geq 0$ on $\mathbb{R}$ and the two Cauchy data u_0 and u_1 are supported in $\mathbb{R}_+$.

Define for each t

$$E_{\pm}(t) := \frac{1}{2} \int_{\mathbb{R}^{\pm}} \left(\left(\frac{\partial u}{\partial t}(t,x) \right)^2 + c^2 \left\{ \left(\frac{\partial u}{\partial x}(t,x) \right)^2 + V(x)(u(t,x))^2 \right\} \right) dx.$$

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Then $E(t) = E_-(t) + E_+(t)$ and conservation of E through time implies

$$\forall\, t \geq 0, \quad E_+(t) \leq E(t) = E(0) = E_+(0).$$

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Then $E(t) = E_-(t) + E_+(t)$ and conservation of E through time implies

$$\forall\, t \geq 0, \quad E_+(t) \leq E(t) = E(0) = E_+(0).$$

Consequently, the local energy evaluated on $\mathbb{R}_+$ is always smaller than the initial one.

Motivation

Assumptions

V changes its sign on $\mathbb{R}$ ($V \geq 0$ on $\mathbb{R}^+$, $V < 0$ on $\mathbb{R}^-$) and Cauchy data u_0 and u_1 are still supported in $\mathbb{R}_+$.

Then, conservation of E implies that if for some $t \geq 0$,

$$E_-(t) = \frac{1}{2}\int_{\mathbb{R}^-}\left(\left(\frac{\partial u}{\partial t}(t,x)\right)^2 + c^2\left\{\left(\frac{\partial u}{\partial x}(t,x)\right)^2 + V(x)(u(t,x))^2\right\}\right)dx < 0$$

we now have

$$E_+(t) > E_-(t) + E_+(t) = E(t) = E(0) = E_+(0).$$

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$$E_+(t) > E_-(t) + E_+(t) = E(t) = E(0) = E_+(0).$$

Hence, the energy evaluated on $\mathbb{R}_+$ at time t is strictly greater than the one evaluated on $\mathbb{R}^+$ at initial time.

Motivation

Superradiance means that it is possible to get more local energy E_+ than the initial one : the energy gain defined by $g(t) := E_+(t)/E_+(0)$ may become greater than 1. If g can be arbitrarily large, this is hyperradiance.

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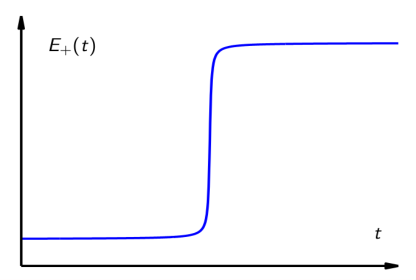
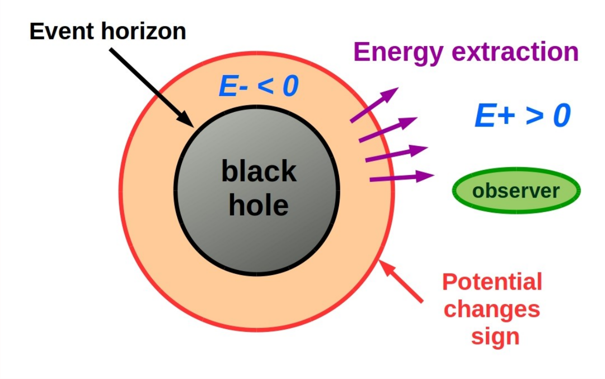
In this case, there is extraction of energy from the left half-line to the right half-line.



Motivation

Roughly speaking, superradiance means that

$$\begin{aligned} 1 &= 0 + 1 \quad \text{with } \frac{2}{1} > 1! \\ &= -1 + 2 \end{aligned}$$



Mathematical model

Unknown function : charged scalar field $\phi = \phi(t, x)$ with charge q, mass m and angular momentum $l \in \mathbb{N}$ $(-\Delta_{S^2}\phi = l(l+1)\phi)$.

Evolution of ϕ outside a Reissner-Nordstrøm black hole is governed by the equation

$$\left\{ \frac{\partial^2}{\partial t^2} - 2i\frac{qQ}{r}\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + F(r)\left(\frac{l(l+1)}{r^2} + m^2 + \frac{F'(r)}{r}\right) - \frac{q^2Q^2}{r^2} \right\}\phi = 0\,,$$

with $Q \neq 0$, $M > |Q|$ charge and mass of the black hole, x quantity computed in terms of r on the form $r = G(x)$ (*Regge-Wheeler* change of coordinate) and

$$F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}.$$

Mathematical model

We set

$$r_+ = M + \sqrt{M^2 - Q^2}$$

as the black hole event horizon.

The black hole exterior is $]r_+, \infty[$ and an exterior observer cannot access the region $]0, r_+]$.

We then have

$$G(x) \simeq \begin{cases} x & \text{if } x \to \infty \\ r_+ & \text{if } x \to -\infty \end{cases}$$

Mathematical model

Conserved quantity :

$$E = \frac{1}{2}\int_{\mathbb{R}} \left\{ |\partial_t \phi|^2 + |\partial_x \phi|^2 + \left(F(r)\left(\frac{l(l+1)}{r^2} + m^2 + \frac{F'}{r} \right) - \frac{q^2 Q^2}{r^2} \right) |\phi|^2 \right\} dx$$

When the black hole is charged ($Q \neq 0$), the potential term

$$\mathcal{V} = F(r)\left(\frac{l(l+1)}{r^2} + m^2 + \frac{F'}{r} \right) - \frac{q^2 Q^2}{r^2}$$

can change sign.

Suggestion : consider in this case Cauchy data supported at the right-hand side domain in order to detect the occurrence of superradiance.

Toy-model

$$(\partial_t - iV(x))^2\phi - c^2\partial_x^2\phi + P(x)\phi = 0,$$

with V and P functions on $\mathbb{R}$. The spatial domain is splitted into $D_- = \{x \leq -L\}$, $D_L = \{-L < x < 0\}$ and $D_+ = \{x \geq 0\}$, where L stands for a smoothing length.

Functions V and P are assumed to be constant on each half-line and continuously connected on the transition interval D_L.

- **Massless case :** $P \equiv 0$, $V = V_0$ on D_-, $V = 0$ on D_+.
- **Massive case :** $P = 0$, $V = V_0$ on D_-, $P = P_0$, $V = 0$ on D_+.
- **Extremal case :** $L = 0$: discontinuous potentials generating hyperradiant modes.

Aim of this study

General goal :

- Construction of a numerical code that enables the simulation of the model in realistic cases, with use of finite differences discretization.
- Simulation of hyperradiance for the toy-model.
- Search for superradiance in the Reissner-Nordstrøm model for a wide variety of parameters.

The problem of boundary conditions

When dealing with a partial differential equation, the model is set on the whole space $\mathbb{R}^n$ (example : wave propagation in vacuum).

$$(\mathcal{P}) \qquad \begin{cases} P(\partial_t, \partial_x)u = 0 & x \in \mathbb{R}^n \quad t > 0 \\ \text{Cauchy data} & x \in \mathbb{R}^n. \end{cases}$$

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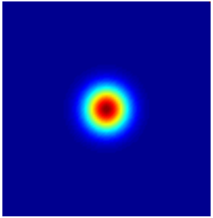
Numerical resolution of $(\mathcal{P})$: use of a numerical method with discretization tools that require to deal with a bounded domain Ω of $\mathbb{R}^n$. We thus need to consider the new problem

$$(\mathcal{P}') \qquad \begin{cases} P(\partial_t, \partial_x)v = 0 & x \in \Omega \quad t > 0 \\ \text{Cauchy data} & x \in \Omega \\ \text{Boundary conditions} & x \in \partial\Omega \quad t \geq 0. \end{cases}$$

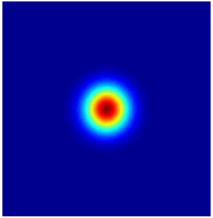
The problem of boundary conditions

Question : which conditions should be set at the boundary Γ of Ω in order to get a "correct" solution v?

Optimal case : $v = u|_\Omega$ (transparent boundary conditions).



On $\mathbb{R}^d$, time $t = 0$



On Ω, time $t = 0$

Major drawback : these conditions are not explicitly given !! We need to find out a strategy to derive them mathematically...

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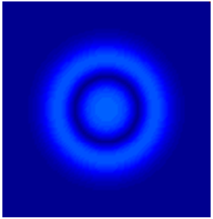
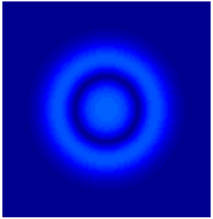


On $\mathbb{R}^d$, time $t = t_1$



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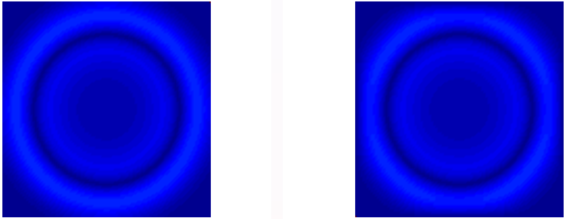

On $\mathbb{R}^d$, time $t = t_2$ On Ω, time $t = t_2$

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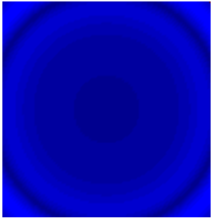
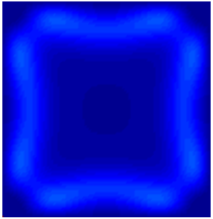


On $\mathbb{R}^d$, time $t = t_3$



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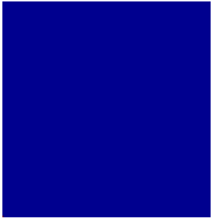
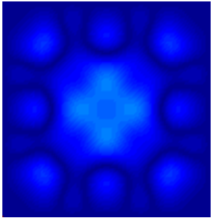


On $\mathbb{R}^d$, time $t = t_4$



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Model problem : the wave equation

Free-potential wave equation in the one-dimensional case :

$$\begin{cases} \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = 0 & x \in \mathbb{R} \quad t > 0 \\ u(0,x) = u_0(x) & x \in \mathbb{R} \\ \dfrac{\partial u}{\partial t}(0,x) = u_1(x) & x \in \mathbb{R} \end{cases}$$

with $c > 0$ and where u_0 are u_1 are given functions of x.

Goal : derivation of natural conditions at the boundary $x = 0$ of the negative half-line $]-\infty, 0[$ in such a way that the corresponding solution will evolve as if the solution was computed on $\mathbb{R}$.

Factorization of the wave operator :

For $\mathcal{C}^2$ functions, one has the identity

$$\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} = \left\{ \frac{\partial}{\partial t} - c\frac{\partial}{\partial x} \right\} \circ \left\{ \frac{\partial}{\partial t} + c\frac{\partial}{\partial x} \right\},$$

with

- $\frac{\partial}{\partial t} - c\frac{\partial}{\partial x} := \mathcal{T}_-$ left-traveling transport operator,
- $\frac{\partial}{\partial t} + c\frac{\partial}{\partial x} := \mathcal{T}_+$ right-traveling transport operator.

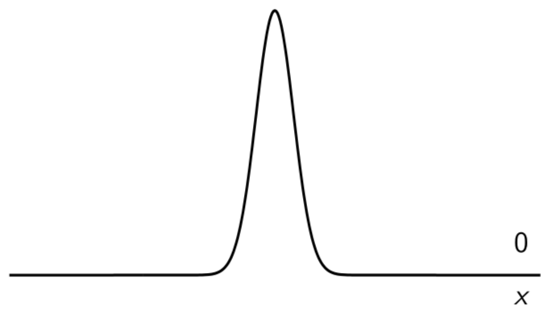
General form of the solutions :

$$u(t,x) = F(x+ct) + G(x-ct),$$

where F and G can be computed in terms of the Cauchy data of the problem : hence, u can be expressed as the sum $U_- + U_+$, where U_- and U_+ respectively satisfy $\mathcal{T}_- U_- = 0$ and $\mathcal{T}_+ U_+ = 0$.

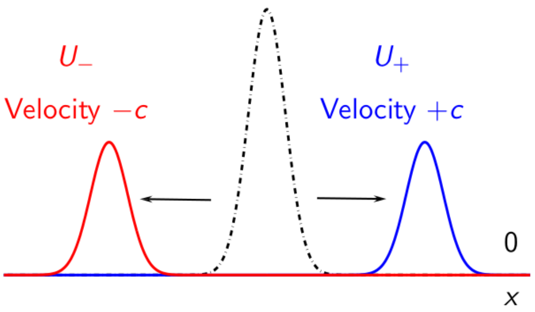
Factorization of the wave operator :

$t = 0$: spatially localized Cauchy data.

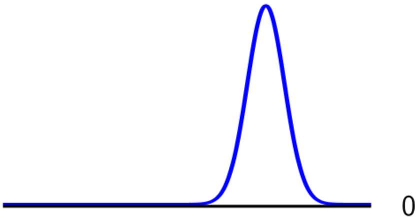


Factorization of the wave operator :

$t > 0$: a part of the solution travels with velocity $+c$ and the other one travels with velocity $-c$.



How to deal with the behaviour at the boundary ?



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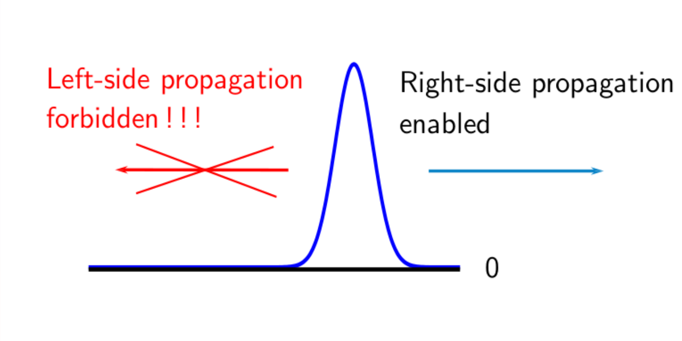

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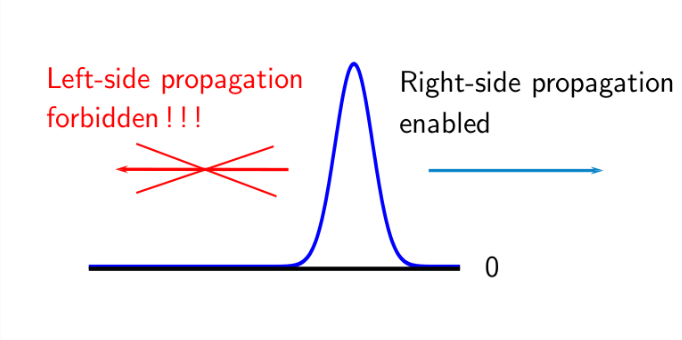

$\Longrightarrow$ We thus set $\mathcal{T}_+ u = \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = 0, \quad t > 0, \quad x = 0.$

Algebraic point of view

The previous factorization of the wave operator is connected to the algebraic identity

$$\tau^2 - c^2\xi^2 = (c\,i\xi - i\tau)(c\,i\xi + i\tau)$$

that holds for the symbol of the wave operator.

Consequently, this boundary conditions can be derived in terms of the time and space Fourier transform. One has to select the right boundary operator.

In this case, the boundary operator is local in time.

Boundary problem

The same considerations can be made when dealing with the positive half-line : this naturally gives the boundary condition

$$\mathcal{T}_- u = \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = 0, \quad t > 0, \quad x = 0.$$

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New problem on $]-\infty, 0[$ **:**

$$\begin{cases} \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = 0 & x \in]-\infty, 0[\quad t > 0 \\ c\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial t} = 0 & x = 0 \quad t > 0 \\ u(0, x) = u_0(x) & x \in]-\infty, 0[\\ \dfrac{\partial u}{\partial t}(0, x) = u_1(x) & x \in]-\infty, 0[. \end{cases}$$

This problem involves a *first-order* boundary condition.

What about the modified problem ?

New equation :

$$\left(\frac{\partial}{\partial t} - iV\right)^2 u - c^2 \frac{\partial^2 u}{\partial x^2} + Pu = 0.$$

The same strategy as before can be made. We finally obtain the transparent condition

$$c\frac{\partial u}{\partial x} + \sqrt{P + \left(\frac{\partial}{\partial t} - iV\right)^2} u = 0, \quad x = 0.$$

that involves a pseudodifferential operator B. Serious drawback : this condition is nonlocal. This can be very costly when dealing with the numerical implementation.

What about the modified problem ?

Numerical trick :

We choose to deal with an extra contribution, say v, which is defined on the whole numerical domain and that handles the nonlocality.

Setting

$$v := \sqrt{P + \left(\frac{\partial}{\partial t} - iV\right)^2}\,\widetilde{u}$$

where $\widetilde{u}(t,x) = u(t,-x)$, it can be found that v solves on the same equation as u. Since v is initially supported on the positive half-line, we have the boundary condition

$$c\frac{\partial v}{\partial x} - \sqrt{P + \left(\frac{\partial}{\partial t} - iV\right)^2}\, v = 0, \quad x = 0.$$

What about the modified problem ?

Boundary conditions

- $c\frac{\partial u}{\partial x} + v = 0$ (local condition)
- $c\frac{\partial v}{\partial x} - Bv = 0.$

Since v is such that $v(t,0) = B\widetilde{u}(t,0) = Bu(t,0)$, the second boundary condition rewrites

$$c\frac{\partial v}{\partial x} - B^2 u = 0,$$

where B^2 is local. This finally gives the local boundary condition

$$c\frac{\partial v}{\partial x} - \left(\frac{\partial}{\partial t} - iV\right)^2 u - Pu = 0.$$

What about the modified problem ?

It is thus found that the original problem involving the transparent condition turns into the locally coupled problem

$$
\begin{cases}
\left(\dfrac{\partial}{\partial t} - iV\right)^2 u - c^2 \dfrac{\partial^2 u}{\partial x^2} + Pu = 0 & x < 0 \quad t > 0 \\
\left(\dfrac{\partial}{\partial t} - iV\right)^2 v - c^2 \dfrac{\partial^2 v}{\partial x^2} + Pv = 0 & x < 0 \quad t > 0 \\
c\dfrac{\partial u}{\partial x} + v = 0 & x = 0 \quad t > 0 \\
c\dfrac{\partial v}{\partial x} - \left(\dfrac{\partial}{\partial t} - iV\right)^2 u - Pu = 0 & x = 0 \quad t > 0 \\
u(t,0) = u_0(x), \qquad v(t,0) = 0 & x \in \mathbb{R} \\
\dfrac{\partial u}{\partial t}(t,0) = u_1(x), \quad \dfrac{\partial v}{\partial t}(t,0) = 0 & x \in \mathbb{R}
\end{cases}
$$

Numerical results

Starting point :

$$(\partial_t - iV(x))^2 u - c^2 \partial_x^2 u + P(x)\phi = 0.$$

The wave equation is written as the first-order system

$$(\partial_t - iV(x)) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ c^2\partial_x^2 - P(x) & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with $w_1 = u$, $w_2 := (\partial_t - iV)u$ and the Cauchy data $w_1(0) = u(0)$, $w_2(0) = \partial_t u(0) - iVu(0)$. We then deal with the problem

$$\partial_t \mathcal{U} + \mathcal{L}\mathcal{U} = 0,$$

where $\mathcal{L} = \mathcal{L}(x, \partial_x)$ stands for a second-order differential operator.

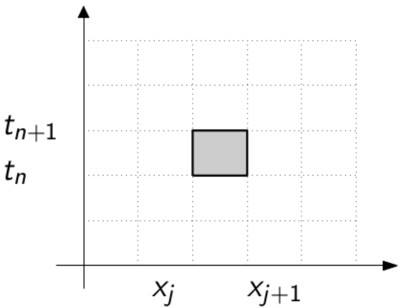
Discretization

Equation : $\partial_t \mathcal{U} + \mathcal{L}\mathcal{U} = 0$

Time and space finite differences discretization, with use of a semi-implicit scheme $\mathcal{U}_j^n = \mathcal{U}(t_n, x_j)$ written at gridpoints $t_n = n\,\delta t$ and $x_j = jh$:

$$\frac{\mathcal{U}_j^{n+1} - \mathcal{U}_j^n}{\delta t} + (\mathcal{L}\mathcal{U}^{n+1/2})_j = 0,$$

with $u^{n+1/2} = (u^n + u^{n+1})/2$: linear system at each time iteration.



Space discretization of the problem is performed on J intervals and transparent conditions are implemented at the boundary.

Energy conservation

We aim to compute a approximate solution that preserves energy conservation at a discrete level.

Discrete energy :

$$E_n = \frac{1}{2} \sum_{j \in \mathbb{Z}} \left(|v_j^n|^2 + \left| \frac{u_{j+1}^n - u_j^n}{h} \right|^2 + P_j |u_j^n|^2 \right) + \sum_{j \in \mathbb{Z}} \Im(V_j \bar{u}_j^n v_j^n)$$

(consistent with continuous energy).

Theorem

Using the semi-implicit discretization on the whole space, one has $E_n = E_0, \quad \forall\, n \geq 0$.

Validation

Model problem :

$$(\partial_t - iV)^2\phi - c^2\partial_x^2\phi + P\phi = 0,$$

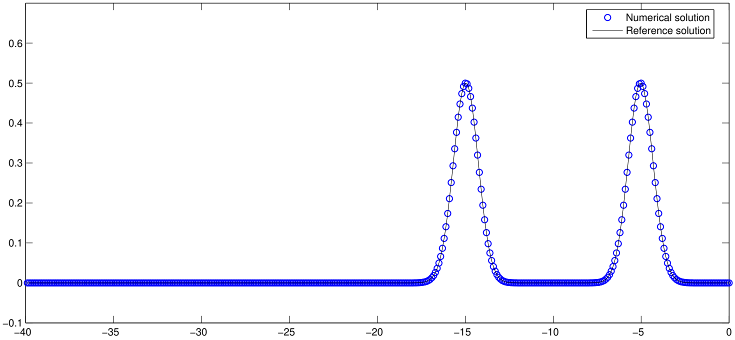
where V and P constant.

Test with a Gaussian Cauchy data such that two opposite waves are generated :

$$\phi(0,x) = e^{-x^2}; \quad \partial_t\phi(0,x) = 0.$$

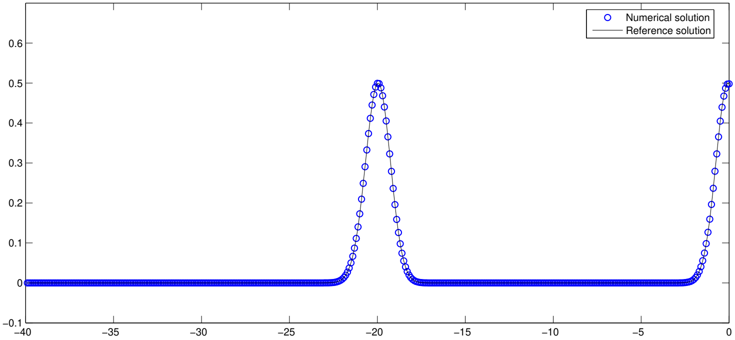
Numerical test : influence of the boundary condition on the solution and comparison with a *reference solution* computed on a larger domain.

Validation, case $V = 0$, $P = 0$



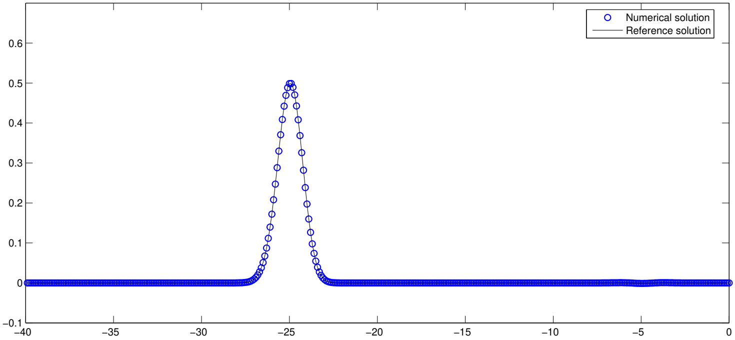
$t = 5$

Validation, case $V = 0$, $P = 0$



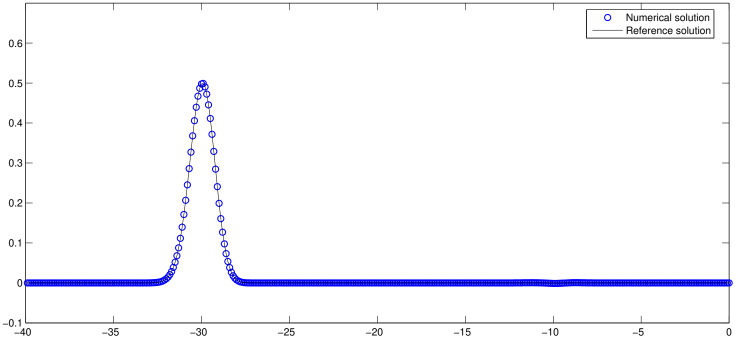
$t = 10$

Validation, case $V = 0$, $P = 0$



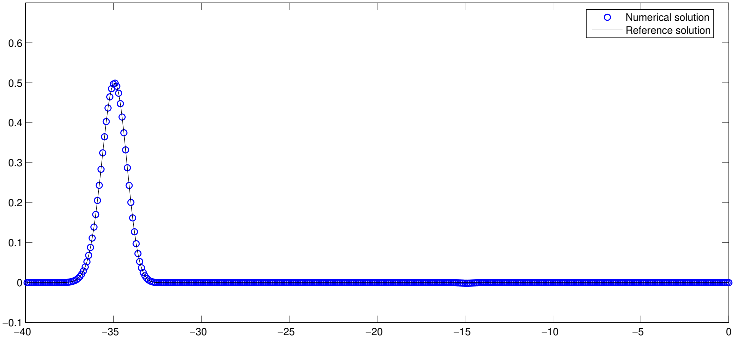
$t = 15$

Validation, case $V = 0$, $P = 0$



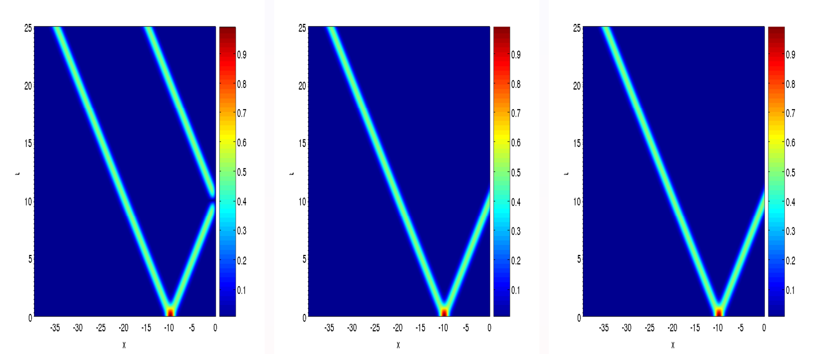
$t = 20$

Validation, case $V = 0$, $P = 0$



$t = 25$

Validation, case $V = 0$, $P = 0$

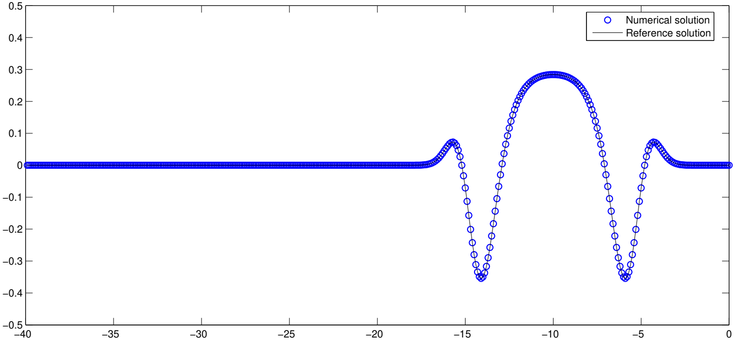


Dirichlet conditions

Transparent conditions

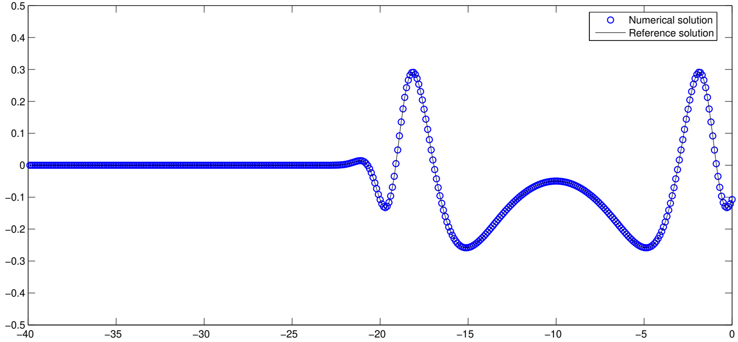
Reference solution

Validation, case $V = 0$, $P = 1$



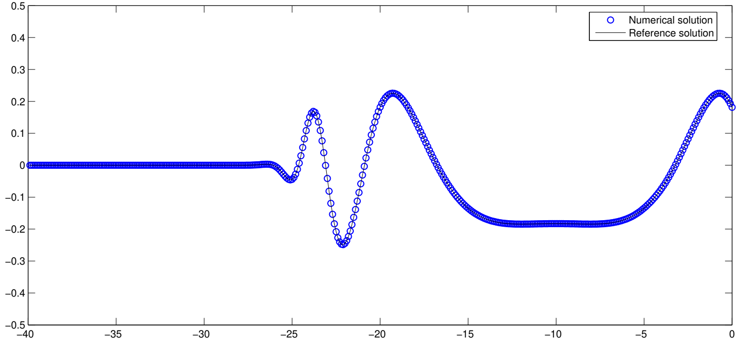
$t = 5$

Validation, case $V = 0$, $P = 1$



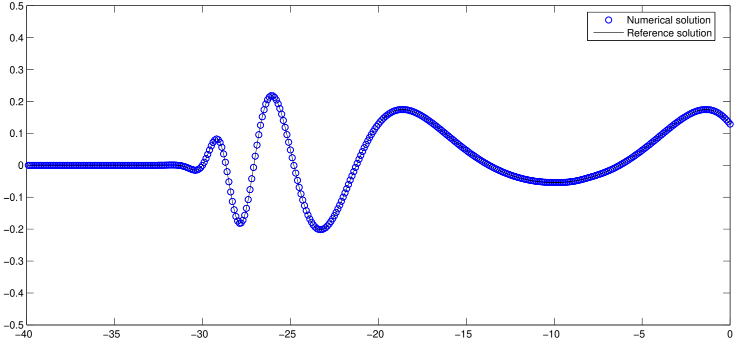
$t = 10$

Validation, case $V = 0$, $P = 1$



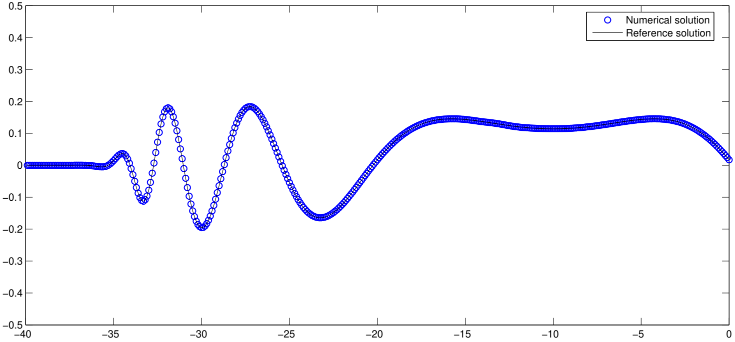
$t = 15$

Validation, case $V = 0$, $P = 1$



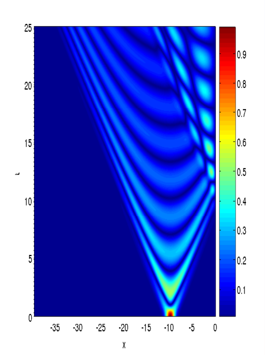
$t = 20$

Validation, case $V = 0$, $P = 1$

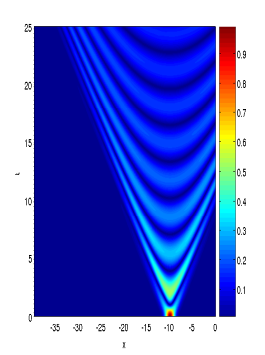


$t = 25$

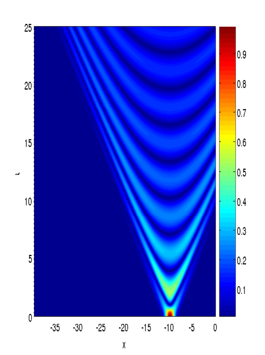
Validation, case $V = 0$, $P = 1$



Dirichlet conditions

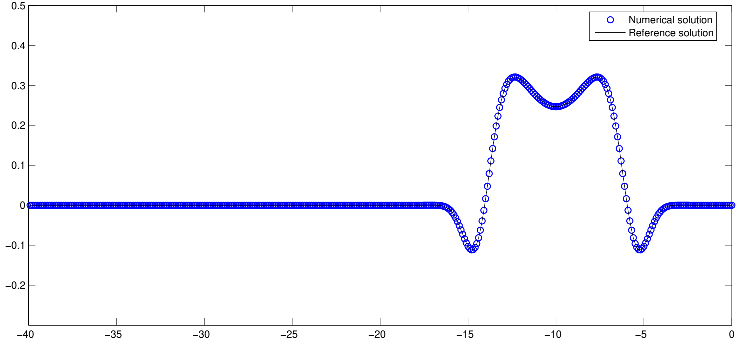


Transparent conditions



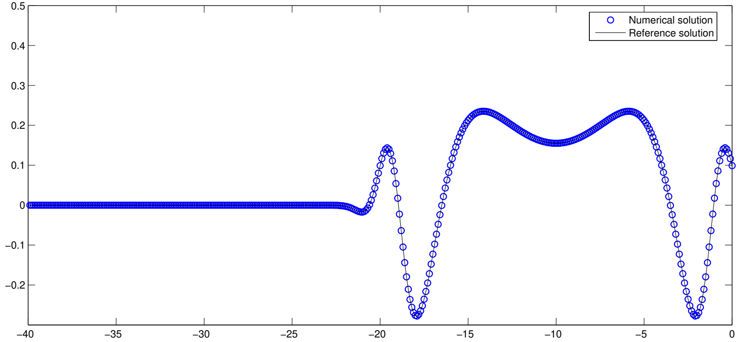
Reference solution

Validation, case $V = 1$, $P = 1$



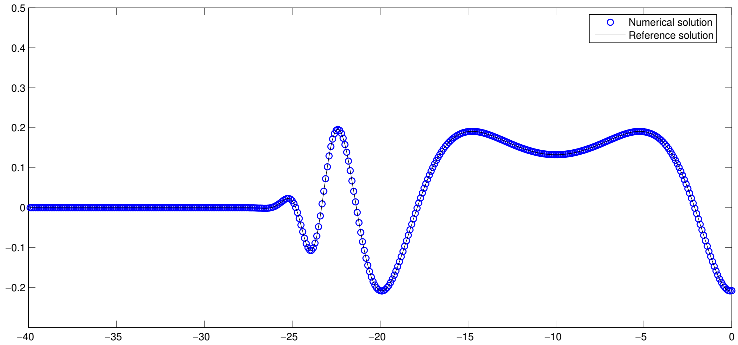
$t = 5$

Validation, case $V = 1$, $P = 1$



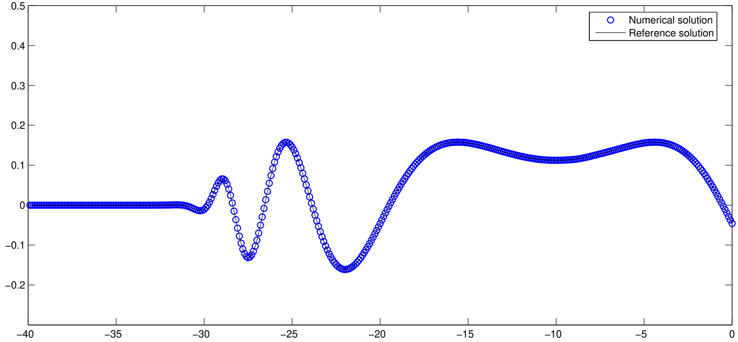
$t = 10$

Validation, case $V = 1$, $P = 1$



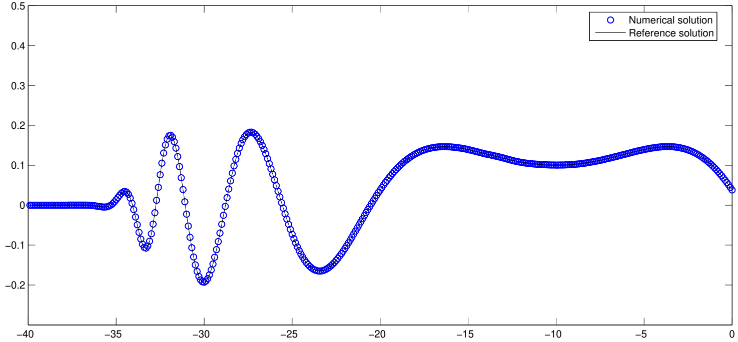
$t = 15$

Validation, case $V = 1$, $P = 1$



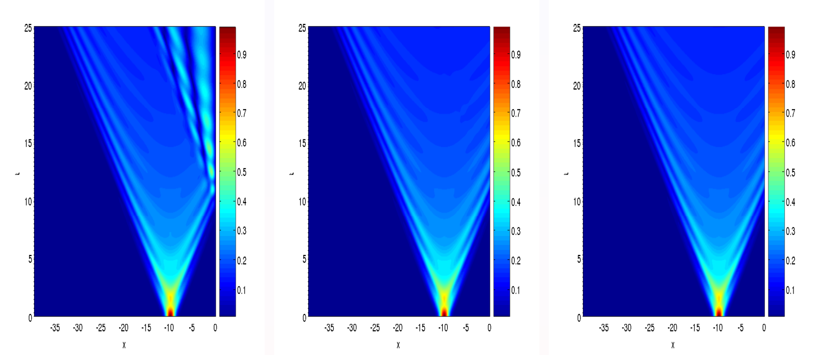
$t = 20$

Validation, case $V = 1$, $P = 1$



$t = 25$

Validation, case $V = 1$, $P = 1$



Dirichlet conditions

Transparent conditions

Reference solution

Hyperradiance for the toy-model

Model problem :

$$(\partial_t - iV(x))^2\phi - c^2\partial_x^2\phi + P(x)\phi = 0,$$

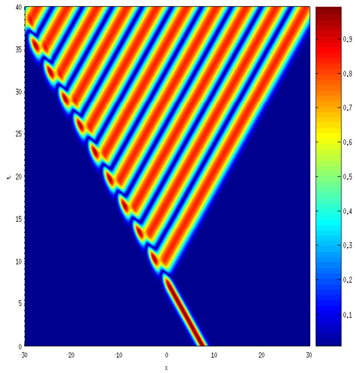
in the massless case $P = 0$, $V = 1$ if $x \leq -L$, $V = 0$ if $x \geq 0$, with V decreasing and smooth between $-L$ and 0.

Test with a Gaussian Cauchy data that mimics a left-side propagation, solving the free transport equation $\partial_t\phi - c\partial_x\phi = 0$ ("left-sided" missile) :

$$\phi(0,x) = e^{-(x-x_0)^2}; \quad \partial_t\phi(0,x) = -2c(x-x_0)e^{-(x-x_0)^2}.$$

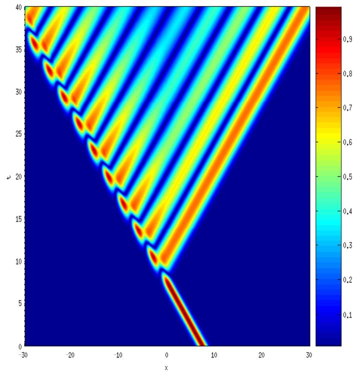
Goal of the simulations : observe the influence of the smoothing length L on the gain of energy, starting from an initial data centered at $x_0 > 0$.

Toy-model : hyperradiance and smoothing



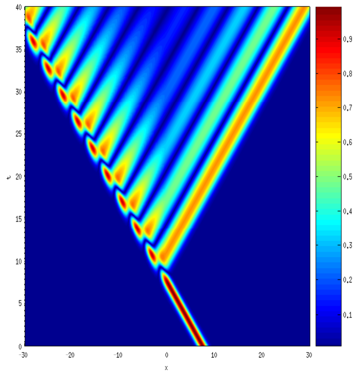
$|\phi(t,x)|$ versus t and x, $L=0$

Toy-model : hyperradiance and smoothing



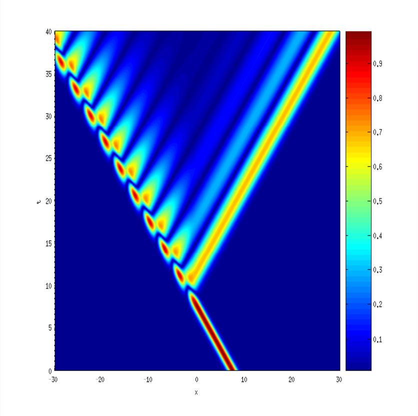
$|\phi(t,x)|$ versus t and x, $L = 0.5$

Toy-model : hyperradiance and smoothing



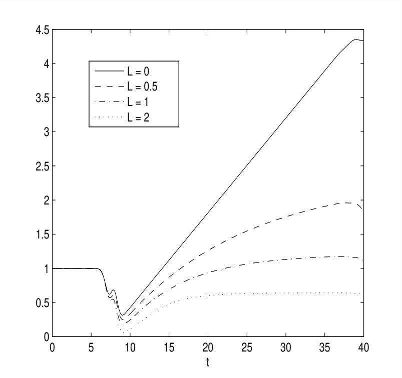
$|\phi(t,x)|$ versus t and x, $L = 1$

Toy-model : hyperradiance and smoothing

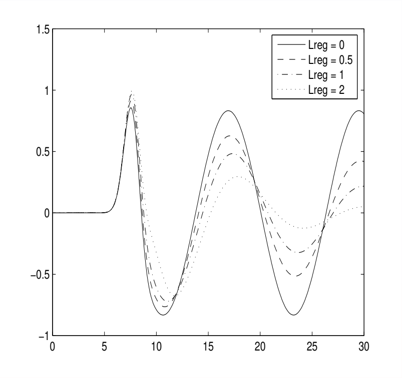


$|\phi(t,x)|$ versus t and x, $L=2$

Toy-model : hyperradiance and smoothing



Energy gain



Trace at $x = 0$

Suggestion

These results show that even with well-adapted conditions, computation of energy gain in terms of spatial integrals cannot be compatible with long time simulations.

The solution will spread out outside the computational domain and larger amount of energy will be neglected.

Suggestion :

Measurement of the energy gain in terms of the flux evaluated on a temporal hypersurface.

Computation of the energy flux

Principle : integration of PDE multiplied with $\partial\overline{\phi}/\partial t$ on $[a, b] \times [t_1, t_2]$: it gives the local energy $E_{a,b}$ between a and b such that

$$E_{a,b}(t_2) - E_{a,b}(t_1) = c^2 \int_{t_1}^{t_2} \left\{ \frac{\partial\overline{\phi}}{\partial t}(t, b)\frac{\partial\phi}{\partial x}(t, b) - \frac{\partial\overline{\phi}}{\partial t}(t, a)\frac{\partial\phi}{\partial x}(t, a) \right\} dx.$$

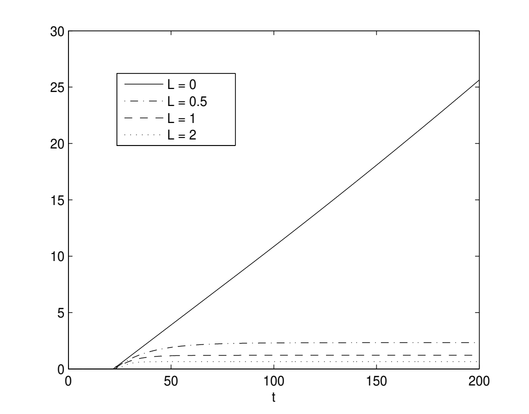
Computation of the energy flux on the surface $x = x_0$:

$$\mathcal{F}_{t_1,t_2}(x_0) = \mathrm{Re}\left(c^2 \int_{t_1}^{t_2} \frac{\partial\overline{\phi}}{\partial t}(t, x_0)\frac{\partial\phi}{\partial x}(t, x_0)\, dx \right).$$

It allows the computation of the gain $g(t)$ evaluating the flux accross the surface :

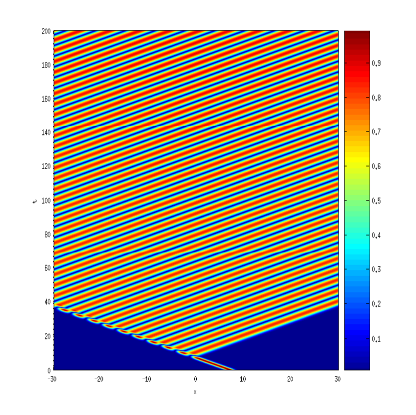
$$g(t) = \frac{E^+(t) + \mathcal{F}_{0,t}(x_0)}{E^+(0)}, \quad x_0 \text{ large.}$$

Computation of the gain with the energy flux



Gain between $t = 0$ and $t = 200$

Long-time behaviour



$|\phi(t,x)|$ versus t and x, $L=0$

Reissner-Nordstrøm model

Aim :

Following the idea of the previous model, we perform simulations leading us to cases where the energy gain of the computed solution becomes greater than 1.

Parametric study with respect to m, M, q, Q, l and Cauchy data $(\phi(0), \partial_t \phi(0))$: difficult problem !

Localized Cauchy data : "left-sided" missiles (as for the toy-model) or

$$\left(\phi(0,x), \frac{\partial \phi}{\partial t}(0,x)\right) = \left(0, e^{-\alpha(x-x_0)^2}\right), \quad x \in \mathbb{R} \quad (\text{"flare"})$$

(giving a positive initial energy $E^+(0)$).

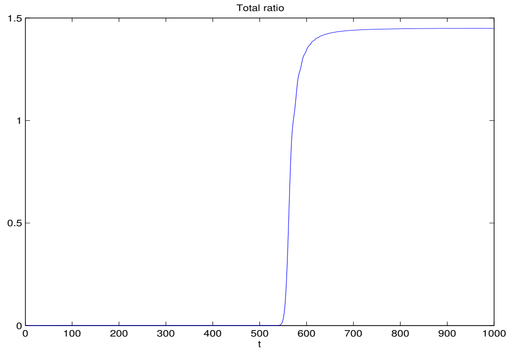
Reissner-Nordstrøm model

Numerical results :

- Superradiance is difficult to detect : it manifests itself for a restricted class of Cauchy data and does not seem stable under perturbations.
- It appears as a low energy phenomenon : simulations performed for $l \gg 1$ exhibit smaller gains.
- It is also a low frequency phenomenon : modulating the Cauchy data with the phase term $e^{i\omega x}$, oscillations will globally make the gain decrease.

Superradiance occurrence

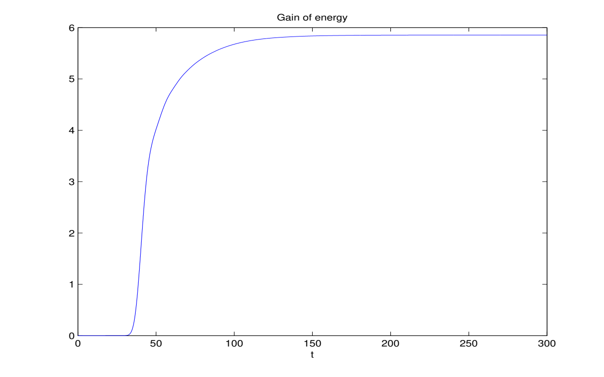
Parameters : $M = 2.1$, $m = 0.1$, $Q = 2$, $q = 1$, $l = 0$, incoming wave packet Cauchy data.



Gain between $t = 0$ and $t = 1000$

Superradiance occurrence

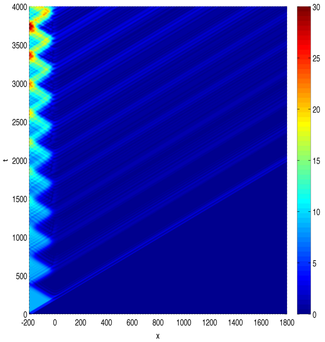
Parameters : $M = 2.1$, $m = 0.1$, $Q = 2$, $q = 1$, $l = 0$, flare Cauchy data.



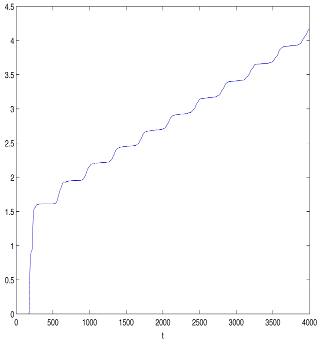
Gain between $t = 0$ and $t = 300$

Hyperradiance

Idea : put mirrors around superradiance domain. Huge amplification of the gain occurs !



Amplitude



Log of energy gain

Conclusion and future work

Results :

- Reliable code with well-adapted boundary conditions and energy computation for long-time calculations.
- Simulation of hyperradiance for the toy-model and superradiance for regularized model.
- Superradiance evidence for the Reissner-Nordstrøm model.

Future work :

- Determination of a superradiant regime which could be independent of a large part of parameters involved in the model.
- Simulation of the nonlinear model.