Classical strings in the Kerr-AdS_5 black hole and holographic observables

based on works with

Irina Aref'eva (MIAN, Moscow), Eric Gourgoulhon(LUTh, Paris),

Marina Usova (MIAN,Moscow), Nikita Tsegelnik(JINR, Dubna)

JHEP 4 (2021) 169; Phys.Part.Nucl. 51 (2020) 4, 535-539; Nucl.Phys.B 979 (2022) 115786; 2211.11722[hep-th]

Anastasia Golubtsova (BLTP JINR, MIAN)

Mathematical Aspects of Black Hole Theory

LUTh, Meudon

December 14, 2022

Outline

Holographic duality

Classical gravity in a $d+1$ curved spacetimes can describe conformal gauge field theoriest at strongly coupling in a d-dimensional spacetime.

Example: The AdS/CFT correspondence (Maldacena'97): classical string theory in AdS_5 with a string length $\ell_s = \sqrt{\alpha'}$ and coupling constant g_s with the radius ℓ is dynamically equivalent to $4d$ $\mathcal{N}=4$ SYM with $SU(N)$ (actually, conformal gauge supersymmetric theory), which "lives" on the boundary of AdS_5.

$$g_{YM}^2 = 2\pi g_s, \quad 2g_{YM}^2 N = \frac{\ell^4}{\alpha'^2}, \quad \lambda = g_{YM}^2 N.$$

$$\boxed{\mathcal{S} = \frac{1}{2\kappa^2}\int d^5x\sqrt{-g}\left(R_5 - 2\Lambda\right), R_{\mu\nu} = \frac{\Lambda}{3}g_{\mu\nu}, \quad \Lambda = -6\ell^2.}$$

Solutions: $ds^2 = r^2(-dt^2 + \delta_{ij}dx^i dx^j) + \frac{dr^2}{r^2}$ AdS with isometry group $SO(2,4)$.

$$\boxed{\left\langle e^{\int d^4x \varphi_0(\vec{x})\mathcal{O}(\vec{x})}\right\rangle_{CFT} = e^{S_{AdS}[\varphi(\vec{x},z)|_{r=\infty} = \varphi_0(\vec{x})]}}$$

Holographic duality

- Pure $AdS_5 \Leftrightarrow T = 0$ 4d $\mathcal{N} = 4$ SYM at strong coupling with $SU(N)$ (Maldacena'97)
  - the isometry group $SO(2,4)$ of AdS_5 is a symmetry group of the dual CFT
  - field theory "lives" on the boundary of the gravity background
  - flat boundary $\Leftrightarrow$ CFT on R^4
  - spherical boundary $\Leftrightarrow$ CFT on cylinder $R \times \mathbb{S}^3$

  Example: global AdS_5

$$ds^2 = -(1 + y^2\ell^2)dT^2 + y^2(d\Theta^2 + \sin^2\Theta d\Phi^2 + \cos^2\Theta d\Psi^2) + \frac{dy^2}{1 + y^2\ell^2}.$$

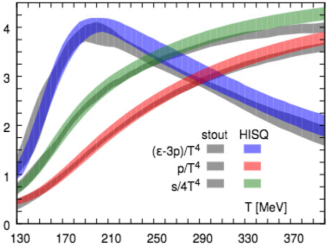
  Boundary: $y \to \infty$, $R \times \mathbb{S}^3$: $ds^2 = -\ell^2 dT^2 + d\Theta^2 + \sin^2\Theta d\Phi^2 + \cos^2\Theta d\Psi^2$.

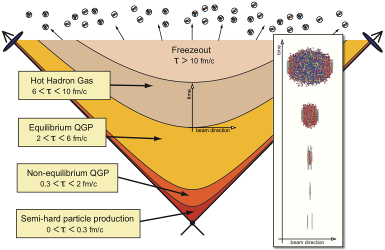
- AdS_5 black hole $\Leftrightarrow$ thermal ensemble of $\mathcal{N} = 4$ SYM at strong coupling with $SU(N)$ (Witten'98)
  T of CFT is identified with the Hawking temperature T_H of black hole

Outline

Quark Gluon Plasma and Holography

- QGP is a deconfined phase of QCD at high T (at strong coupling $\lambda \approx 1$)
- The deconfinement T in the non-rotating case($\mu = 0$) is $T_c \approx 170$ MeV.
- Moreover, QCD at high T has a quasi-conformal behaviour ($T^\mu_\mu = 0$) (lattice)





- The viscosity-to-entropy ratio for QGP from holography

$$\frac{\eta}{s} = \frac{1}{4\pi}.$$

Policastro, Son, Starinets, Phys.Rev.Lett,2001

checked on RHIC I. Arsene et al. [BRAHMS Collaboration],Nucl. Phys. A 757,1, 2005.

Rotating quark-gluon plasma

- It is produced in non-central heavy-ion collisions.
- There is a nonzero total angular momentum related to colliding nuclei.
- The angular momentum remains in the QGP (conserved in time).

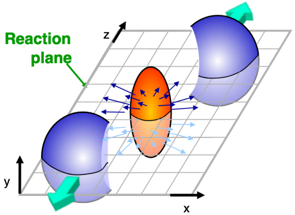


Figure: Geometry of non-central heavy ion collision(Pic. from B.Muller `arXiv:1309.7616`)

The measurements of the Λ, $\bar{\Lambda}$ hyperon polarization by STAR predict $\Omega \sim 6 \pm 1$ MeV. L. Adamczyk et al. (STAR Collaboration),*Nature* 548, 62 (2017).
The value of Ω obtained in the hydrodynamic simulations is $\Omega \sim 20 - 40$ MeV. Y. Jiang, Z.-W. Lin, J. Liao, *Phys. Rev.*C 94, 044910 (2016).
M. Baznat, K. Gudima, A. Sorin and O.Teryaev, Phys. Rev. C 93, 031902(2016).

Rotating quark-gluon plasma

Hawking et.al.'98-99: d rotating CFT $\cong d+1$ Kerr-AdS black holes $(d=3,4,5)$

Rotating QGP can be described by 5d Kerr-AdS (rotating) black holes with an AdS asymptotics

Bhattacharyya et.al.'08: Holographic fluid/gravity correspondence for rotating black holes.

Nata Atmaja and Schalm'10: Rotating QGP $\cong$ Kerr-AdS black hole $(d=4)$

Romaschke et.al.'19: Heavy ion collisions and Kerr-AdS black holes

X. Chen et.al, Gluodynamics and deconfinement phase transition under rotation from holography, 2010.14478 (boosted black hole)

Kerr-AdS_5 and holography

- Bhattacharyya et.al.'08: Holographic fluid/gravity correspondence for rotating black holes.
- Nata Atmaja and Schalm'10: Rotating QGP $\cong$ Kerr-AdS black hole ($d = 4$)
- Romaschke et.al.'19:Heavy ion collisions and Kerr-AdS black holes
- X. Chen et.al, Gluodynamics and deconfinement phase transition under rotation from holography, 2010.14478 (boosted black hole)
- Garbiso and Kaminski, "Hydrodynamics of simply spinning black holes & hydrodynamics for spinning quantum fluids," *JHEP* **12** (2020), 112.
- V. Cardoso, et.al., Holographic thermalization, quasinormal modes and superradiance in Kerr-AdS, JHEP 04(2014)183.
- J.B. Amado et.al., *On the Kerr-AdS/CFT correspondence*, JHEP 08(2017)094.
- M. Cvetič et. al., *BPS Kerr-AdS time machines*, JHEP 07(2018)088 .
- A. Castro et.al., *$5D$ rotating black holes and the nAdS$_2$/nCFT$_1$ correspondence*, JHEP10(2018)042.
- M. David et.al.,"Logarithmic Corrections to the Entropy of Rotating Black Holes and Black Strings in AdS$_5$," [arXiv:2106.09730 [hep-th]].
- O. Aharony et. al., "A gravity interpretation for the Bethe Ansatz expansion of the $\mathcal{N} = 4$ SYM index," [arXiv:2104.13932 [hep-th]].

Outline

5D Kerr-AdS in Boyer-Lindquist coordinates

Rotating-at-infinity frame   Hawking et al.'98

$$
\begin{aligned}
ds^2 &= -\frac{\Delta_r}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi_a}d\phi - \frac{b\cos^2\theta}{\Xi_b}d\psi\right)^2 + \\
&+ \frac{\Delta_\theta\sin^2\theta}{\rho^2}\left(adt - \frac{(r^2+a^2)}{\Xi_a}d\phi\right)^2 + \\
&+ \frac{\Delta_\theta\cos^2\theta}{\rho^2}\left(bdt - \frac{(r^2+b^2)}{\Xi_b}d\psi\right)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \\
&+ \frac{(1+r^2\ell^2)}{r^2\rho^2}\left(abdt - \frac{b(r^2+a^2)\sin^2\theta}{\Xi_a}d\phi - \frac{a(r^2+b^2)\cos^2\theta}{\Xi_b}d\psi\right)^2,
\end{aligned}
$$

where Hopf coordinates $0 \le \phi, \psi \le 2\pi$, $0 \le \theta \le \pi/2, M$ is the mass of BH.

$$
\begin{aligned}
\Delta_r &= \frac{1}{r^2}(r^2+a^2)(r^2+b^2)(1+r^2\ell^2) - 2M, \Delta_\theta = (1 - a^2\ell^2\cos^2\theta - b^2\ell^2\sin^2\theta), \\
\rho^2 &= (r^2 + a^2\cos^2\theta + b^2\sin^2\theta), \quad \Xi_a = (1-a^2\ell^2), \quad \Xi_b = (1-b^2\ell^2).
\end{aligned}
$$

1. $a \neq 0, b \neq 0$
2. $a = b$
3. $a \neq 0, b = 0$

The Hawking temperature (Gibbons et.al.'2004)

$$T_H = \frac{1}{2\pi}\left(r_+(1+r_+^2\ell^2)\Big(\frac{1}{r_+^2+a^2}+\frac{1}{r_+^2+b^2}\Big)-\frac{1}{r_+}\right).$$

r_+ – the outer horizon, defined as a greater solution to

$$\frac{1}{r^2}(r^2+a^2)(r^2+b^2)(1+r^2\ell^2)-2M=0.$$

The angular momenta

$$J_a = \frac{\pi m a}{2\Xi_a^2\Xi_b}, \quad J_b = \frac{\pi m b}{2\Xi_b^2\Xi_a}.$$

In Boyer-Lindquist coordinates the Kerr-AdS metric is asymptotic to AdS_5 in a rotating frame with angular velocities

$$\Omega_a^\infty = -a\ell^2, \quad \Omega_b^\infty = -b\ell^2.$$

The boundary metric $r \to +\infty$:

$$ds_{BL}^2 = -dt^2 + \frac{2a\sin^2\theta}{\Xi_a}dtd\phi + \frac{2b\cos^2\theta}{\Xi_b}dtd\psi + \frac{\ell^2}{\Delta_\theta}d\theta^2 + \frac{\ell^2\sin^2\theta}{\Xi_a}d\phi^2 + \frac{\ell^2\cos^2\theta}{\Xi_b}d\psi^2.$$

The transformations from Boyer-Lindquist coordinates to asymptotically AdS coordinates are

$$\begin{aligned} \Xi_a y^2 \sin^2\Theta &= (r^2+a^2)\sin^2\theta, \quad \Xi_b y^2\cos^2\Theta = (r^2+b^2)\cos^2\theta, \\ \Phi &= \phi + a\ell^2 t, \quad \Psi = \psi + b\ell^2 t, \quad T = t, \quad \Delta = 1 - a^2\sin^2\Theta - b^2\cos^2\Theta. \end{aligned}$$

For $a=b$ 5d Kerr-AdS metric in AdS coordinates

Static-at-infinity frame

$$\begin{aligned} ds^2 &= -(1+y^2\ell^2)dT^2 + y^2(d\Theta^2 + \sin^2\Theta d\Phi^2 + \cos^2\Theta d\Psi^2) + \\ &\quad \frac{2M}{y^2\Xi^3}(dT - a\sin^2\Theta d\Phi - a\cos^2\Theta d\Psi)^2 + \frac{y^4 dy^2}{y^4(1+y^2\ell^2) - (2M/\Xi^2)y^2 + (2Ma^2/\Xi^3)}. \end{aligned}$$

The 5d Kerr-AdS black hole with $a \neq b$ near the boundary Gibbons'04

$$\begin{aligned} ds^2 &\simeq -(1+y^2)dT^2 + \frac{dy^2}{1+y^2-\frac{2M}{\Delta^2 y^2}} + y^2(d\Theta^2 + \sin^2\Theta d\Phi^2 + \cos^2\Theta d\Psi^2) \\ &+ \frac{2M}{\Delta^3 y^2}dT^2 + \frac{2Ma^2\sin^4\Theta}{\Delta^3 y^2}d\Phi^2 + \frac{2Mb^2\cos^4\Theta}{\Delta^3 y^2}d\Psi^2 - \\ &- \frac{4Ma\sin^2\Theta}{\Delta^3 y^2}dTd\Phi - \frac{4Mb\cos^2\Theta}{\Delta^3 y^2}dTd\Psi + \frac{4Mab\sin^2\Theta\cos^2\Theta}{\Delta^3 y^2}d\Phi d\Psi, \end{aligned}$$

The conformal boundary of 5d Kerr-AdS is 4d $R\times S^3$ at $y\to\infty$:

$$ds^2 = -dT^2 + d\Theta^2 + \sin^2\Theta d\Phi^2 + \cos^2\Theta d\Psi^2.$$

Thermodynamics of 5d Kerr-AdS

- The black holes have the Hawking temperature (T_H), entropy (S) and the free energy (F), which are associated to the corresponding quantities of the dual theory
- AdS black holes have the Hawking-Page phase transition (a non-trivial dependence of the Hawking temperature on the location of the horizon), that is actually the first order phase transition.
- The entropy of the Kerr-AdS black hole is

$$S = \frac{A}{4G} = \frac{\pi^2}{2\Xi_a \cdot \Xi_b} \frac{(r_+^2 + a^2)(r_+^2 + b^2)}{r_+},$$

  where $\Xi_a = 1 - a^2\ell^2$ and $\Xi_b = 1 - a^2\ell^2$, r_+ – is the outer horizon.
- The free energy

$$F(r) = \int_r^{r_+} S(x) T_H'(x) dx.$$

The dependence of T on r_+

Aref'eva, AG, Gourgoulghon'JHEP 4 (2021)

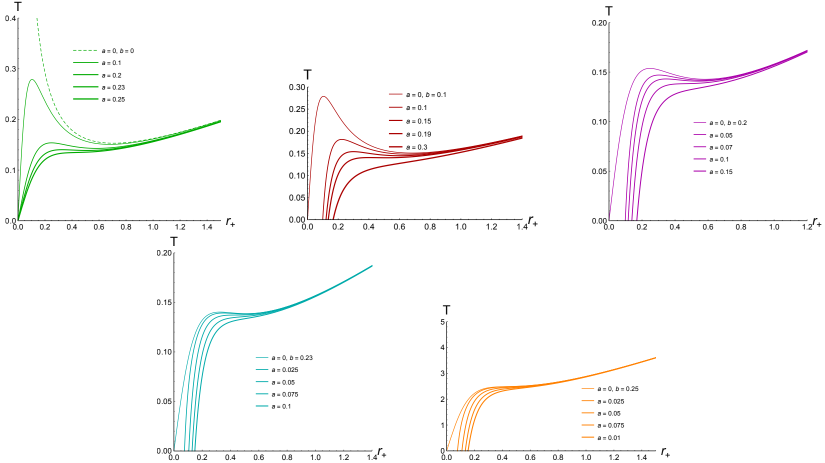


Figure: The plots show the dependence of temperature on r_+ for various values of a and b.

The dependence of F on T

Aref'eva, AG, Gourgoulghon'JHEP 4 (2021)

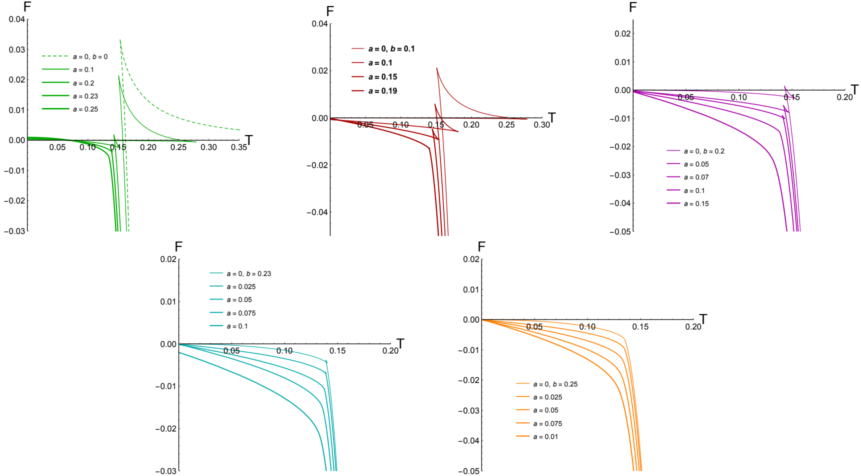


FIGURE: A logarithmic dependence of free energy on temperature on various values of a and b.

Locations of the first order phase transition

Aref'eva, AG, Gourgoulghon'JHEP 4 (2021)

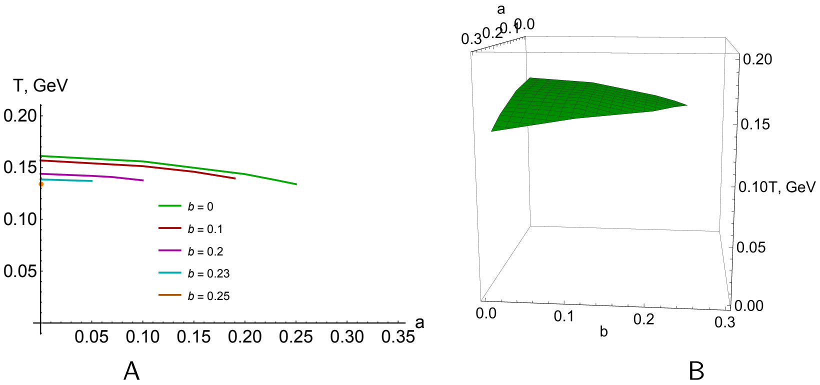


Figure: Locations of the first order phase transition in terms of a and b. A) Locations of the end points for fixed b are indicated by dots. We see that all end points have the same temperature $T_{\rm CEP} = 0.134$ GeV. B) The location of the phase transition $T_{cr} = T_{cr}(a^2, b^2)$ is shown by the green surface. ℓ =1, $[a] \sim$ fm, $[b] \sim$ fm, $[\ell] \sim$ fm^{-1}, $[\Omega_{a,b}] \sim [fm]^{-1}$ $\Omega_{a,b} \simeq 19.7 - 45.3 \quad$ MeV

The rotation affects on the phase transition.

Holography

The temperature of the phase transition decreases with the rotation was shown using the holographic approach in the recent work X. Chen et.al. , 2010.14478 .

Effective and lattice models

- Effective models of rotating QGP: Ebihara et.al., Phys. Lett. B **764** (2017), 94-99, Chernodub and Gongyo, JHEP **01** (2017), 136, Chernodub, Phys. Rev. D **103** (2021) no.5, 054027, Fujimoto et.al, 2101.09173, it was found out the rotation decreases the deconfinement temperature.
- Lattice QCD calculations in works [V. V. Braguta, A. Y. Kotov, D. D. Kuznedelev, A. A. Roenko, *JETP Letters* 112 (1) (2020), V.V. Braguta, A.Yu. Kotov, D.D. Kuznedelev, A.A. Roenko, *Phys. Rev.* **D** 103, 094515 (2021)] it has been shown that the phase transition temperature T_c increases with growing Ω.

Outline

Classical strings in Kerr-AdS_5

Open strings in asymptotically AdS_{d+1} spacetimes correspond to observables in dual d-dimensional gauge theories. The endpoint of an open string can be identified with a heavy quark, which is located on the boundary of the spacetime.

String	Observable
An open string with one endpoint on the boundary stretching to the horizon	Conjugate momenta of the string (Energy loss of the quark in the medium (E, drag force))
An open string with two endpoints on the boundary	rectangular Wilson loops (heavy quark potential V_{qq}, jet-quenching parameter $\hat{q}$)

A heavy quark in rotating QGP

Heavy quarks as probes of the strongly coupled QGP

- transport properties of QGP can be understood calculating observables related to heavy quark dynamics
- moving through the QGP a quark looses its energy fast because of strong coupling
- energy losses are observed through jet quenching at experiments on RHIC
- for heavy quarks, energy losses can be studied more clearly, since low-energy strong-interaction effects are masked by quarks' mass.

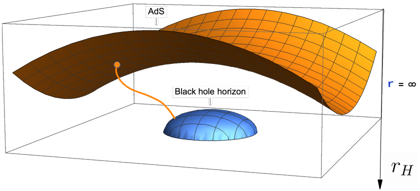
Heavy quarks in AdS/CFT

- The ground states of the open string are the W-bosons and their superpartners of the broken,SU(N), gauge group. Maldacena (9803002)
- $W \to q\bar{q}$
- A heavy quark is related to an open string hanging from the boundary of the holographic background into the bulk

A heavy quark via an open string in AdS BH

S.S. Gubser, Phys. Rev. D74 (2006) 126005; Herzog et.al., JHEP 0607 (2006) 013.

- The external quark "moves" on the boundary of AdS_5 BH along timelike trajectory.
- The string trails out behind the quark, arcing down into AdS_5 BH
- The string exerts a drag force on the external quark
- The quark is infinitely massive



String dynamics through Nambu-Goto action

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int d\sigma^0 d\sigma^1 \sqrt{-g}, \quad g_{\alpha\beta} = G_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu,$$

where $\sigma^0 = t, \sigma^1 = r$; $\theta = \theta(t,r), \phi = \phi(t,r), \psi = \psi(t,r)$; $(G_{\mu\nu})$ –5d Kerr-AdS.

String equations of motion:

$$\frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}\,G_{\mu\nu}\,\partial^\alpha X^\nu\right) - \frac{1}{2}\partial_\mu G_{\rho\nu}\partial_\alpha X^\rho\,\partial^\alpha X^\nu = 0,$$

The conjugate momenta

$$\boxed{\pi^\alpha_\mu = \sqrt{-g}g^{\alpha\beta}G_{\mu\nu}(X)\,\partial_\beta X^\nu}$$

The total momentum charge in the μ direction carried by the string

$$p_\mu = \int d\Sigma_\alpha \pi^\alpha_\mu,$$

where Σ_α is a surface (a line on the worldsheet) on the string worldsheet.

The energy of the string

$$E = -\frac{1}{2\pi\alpha'}\int dr\pi^0_t.$$

Energy of a static quark AG,EG,MU'2021

Arbitrary rotating parameters $a \neq b$. A single quark at rest is described by a static string solution with

$$\theta(r,t) = \theta_0, \quad \phi(t,r) = \phi_0, \quad \psi(t,r) = \psi_0,$$

the conjugate momentum (the energy density of a string) in the BL coordinates

$$\pi^0{}_t = \sqrt{-g} g^{00} G_{tt} = -\left(1 - \frac{\Delta_\theta}{\Delta_r}\left(a^2 \sin^2\theta + b^2\cos^2\theta\right) - \frac{1}{\Delta_r}\frac{a^2 b^2(1+r^2\ell^2)}{r^2}\right).$$

Expanding in small a, then in small b

$$\pi^0{}_t = -1 + \frac{a^2\sin^2\theta}{\ell^2 r^4 - 2M + r^2} + \frac{b^2\cos^2\theta}{\ell^2 r^4 - 2M + r^2}.$$

At the limit $T=0$ that corresponds to $M=0$ we get

$$E\Big|_{T=0} = \frac{1}{2\pi\alpha'}\left(r_m + \ell(a^2\sin^2\theta + b^2\cos^2\theta)\left(\frac{1}{\ell r_m} - \frac{1}{\varepsilon} + \tan^{-1}(\ell r_m)\right)\right),$$

where $\varepsilon = \ell\epsilon$ and $r_H = 0$ with $M=0$ and we define the quark location by r_m .

At $T=0$ the renormalized energy equals to the mass m of the quark.

$$E_{\rm ren}\Big|_{T=0}=\frac{1}{2\pi\alpha'}\left(r_m+\ell(a^2\sin^2\theta+b^2\cos^2\theta)\left(\frac{1}{\ell r_m}+\tan^{-1}(\ell r_m)\right)\right)=m.$$

If one increases the temperature the static thermal mass takes the form

$$M_{\rm rest}=m-\Delta m(T,a,b),$$

where

$$\Delta m(T,a,b)=\frac{\sqrt{\lambda}}{2\pi}\Bigg(\frac{\pi T_H+\sqrt{\pi^2T_H^2-2\ell^2}}{2\ell^2}+\frac{2\ell^4(a^2\sin^2\theta+b^2\cos^2\theta)}{\pi^2T_H^2+\pi T_H\sqrt{\pi^2T_H^2-2\ell^2}}\times$$
$$\Bigg(-\frac{\tanh^{-1}\Big(1+\frac{\epsilon 2\ell^2}{\pi T_H+\sqrt{\pi^2T_H^2-2\ell^2}}\Big)}{\pi T_H+\sqrt{\pi^2T_H^2-2\ell^2}}+\frac{\sqrt{2}\tan^{-1}\Big(\frac{(\pi T_H+\sqrt{\pi^2T_H^2-2\ell^2}+2\epsilon\ell^2)}{\sqrt{2}\sqrt{\pi^2T_H^2+\ell^2+\pi T_H\sqrt{\pi^2T_H^2-2\ell^2}}}\Big)}{\sqrt{\pi^2T_H^2+\ell^2+\pi T_H\sqrt{\pi^2T_H^2-2\ell^2}}}\Bigg)\Bigg),$$

with $\lambda=1/\alpha'^2$.

In the limit of the large T_H

$$\Delta m(T,a,b)\approx\frac{\sqrt{\lambda}}{2}\frac{T_H}{\ell^2},$$

that is in agreement with Herzog et.al.'06

Non-rotating-at-infinity frame (AdS coordinates)

In the limit of zero mass $M = 0$, that corresponds to the zero temperature

$$E|_{T=0} = \frac{1}{2\pi\alpha'}\left(y + 2\left(\frac{M}{y} + M\tan^{-1}y - \frac{i\pi\sqrt{2M}}{4}\right)\left(a^2\sin^2\Theta + b^2\cos^2\Theta\right)\right)\Big|_{y_H+\epsilon}^{y_m} = y_m.$$

This energy corresponds to the rest mass of the quark:

$$E|_{T=0} = \frac{1}{2\pi\alpha'}y_m = m.$$

Finally, we can represent the statice thermal mass of the quark at rest as

$$\Delta m(T,a,b) = \frac{\sqrt{\lambda}}{2\pi}\Bigg(\frac{\pi T_H + \sqrt{\pi^2T_H^2 - 2}}{2} +$$

$$+\Bigg(\frac{2\tanh^{-1}\left(1 + \frac{2\epsilon}{\pi T_H + \sqrt{\pi^2T_H^2-2}}\right)}{\pi T_H + \sqrt{\pi^2T_H^2 - 2}} - \frac{\sqrt{2}\tan^{-1}\left(\frac{1}{\sqrt{2}}\frac{\pi T_H + \sqrt{\pi^2T_H^2-2}+2\epsilon}{\sqrt{\pi^2T_H^2+\pi T_H\sqrt{\pi^2T_H^2-2}+1}}\right)}{\sqrt{\pi^2T_H^2 + \pi T_H\sqrt{\pi^2T_H^2 - 2}+1}}\Bigg)$$

$$\times\frac{\left((\pi^2T_H^2 + \pi T_H\sqrt{\pi^2T_H^2 - 2})^2 - 1\right)(a^2\sin^2\Theta + b^2\cos^2\Theta)}{4(\pi^2T_H^2 + \pi T_H\sqrt{\pi^2T_H^2 - 2})}\Bigg).$$

At high temperature

$$\Delta m(T,a,b) \approx \frac{\sqrt{\lambda}}{2}T_H.$$

The drag force acting on a heavy quark

The drag force

$$\frac{\partial p_\mu}{\partial \sigma^0} = -\frac{1}{2\pi\alpha'}\pi^r_\mu.$$

The curved string solution with small a and b

$a \neq 0, b = 0$, $a = b$ rotating-at-infinity frame, Aref'eva, AG, Gourgoulhon'20

$a \neq b \neq 0$, AG, Gourgoulhon, Usova'21

$$\begin{aligned}
\phi(t,r) &= \Phi_0 + a\ell^2 t + a\phi_1(r) + \mathcal{O}(a^2),\\
\theta(r) &= \Theta_0 + (a+b)^2\theta_1(r) + \mathcal{O}((a+b)^4),\\
\psi(t,r) &= \Psi_0 + b\ell^2 t + b\psi_1(r) + \mathcal{O}(b^2),
\end{aligned}$$

$$\phi_1(r) = \mathfrak{p} \int_{r_+}^{r} \frac{d\bar{r}}{\bar{r}^4 h(\bar{r})}, \quad \psi_1(r) = \mathfrak{q} \int_{r_+}^{r} \frac{d\bar{r}}{\bar{r}^4 h(\bar{r})}, \quad h(r) = \ell^2 + \frac{1}{r^2} - \frac{2M}{r^4},$$

$$\Upsilon' + \frac{2r + 4\ell^2 r^3}{r^2 + \ell^2 r^4 - 2M}\Upsilon + \frac{1 + 4\ell^2 M(n^2-1) - \mathfrak{p}^2 + n^2(\mathfrak{q}^2 - 1) - 3(n^2-1)\ell^4 r^4}{2(1+n)^2(r^2 + \ell^2 r^4 - 2M)^2} \sin(2\Theta_0) = 0,$$

where $n = \frac{b}{a}$ $\Upsilon = \theta_1'$. The derivative of θ_1 can be represented as

$$\begin{aligned}
\theta_1' \quad &= \quad \frac{C_1}{r^4 f(r)} - \frac{3\ell^2(1-n^2) r \sin(2\Theta_0)}{2(1+n)^2 r^4 f(r)} + \\
&+ \quad \frac{(1-n^2)\left(\ell^2 r_H^2 - 1\right)^2 - \mathfrak{p}^2 + n^2 \mathfrak{q}^2}{2(n+1)^2 f(r) r^4 (2\ell^2 r_H^2 + 1) r_H} \tanh^{-1}\left(\frac{r_H}{r}\right) \sin(2\Theta_0) \\
&+ \quad \ell \frac{(1-n^2)\left(\ell^2 r_H^2 + 2\right)^2 - \mathfrak{p}^2 + n^2 \mathfrak{q}^2}{2(n+1)^2 r^4 f(r)(2\ell^2 r_H^2 + 1)\sqrt{\ell^2 r_H^2 + 1}} \tan^{-1}\left(\frac{\ell r}{\sqrt{\ell^2 r_H^2 + 1}}\right) \sin(2\Theta_0),
\end{aligned}$$

r_H is a zero of $h(r)$. The conjugate momenta are

$$\pi_\theta^r = h(r) r^4 \theta_1' a^2 + \mathcal{O}(a^4), \ \pi_\phi^r = h(r) r^4 \phi_1' a + \mathcal{O}(a^2), \ \pi_\psi^r = h(r) r^4 \psi_1' a + \mathcal{O}(a^2).$$

The components of the drag force $a \neq b \neq 0$

$$
\begin{aligned}
\frac{dp_\theta}{dt} &= \Big(-\frac{2C_1}{\sin(2\Theta_0)} + 3r\ell^2\frac{a-b}{a+b} + \frac{3}{r}\frac{a-b}{a+b} \\
&- \frac{\pi\ell}{2(2\ell^2 r_H^2+1)}\frac{\left(a^2-b^2\right)\left(\ell^2 r_H^2+2\right)^2 - a^2\mathfrak{p}^2 + b^2\mathfrak{q}^2}{(a+b)^2\sqrt{\ell^2 r_H^2+1}}\Big)\frac{\sin(2\Theta_0)}{4\pi\alpha'}(a+b)^2 + \mathcal{O}((a+b)^4), \\
\frac{dp_\phi}{dt} &= -\frac{1}{2\pi\alpha'}\mathfrak{p}\sin(\Theta_0)^2 a + \mathcal{O}(a^2), \qquad \frac{dp_\psi}{dt} = -\frac{1}{2\pi\alpha'}\mathfrak{q}\cos(\Theta_0)^2 b + \mathcal{O}(b^2).
\end{aligned}
$$

The component $\frac{dp_\theta}{dt}$ has a linearly divergent term with $r \to \infty$ and is associated to an infinite mass of the heavy quark. In terms of T_H we have

$$
\frac{dp_\theta}{dt} = 3\frac{a-b}{a+b}\left(\ell^2 m_{\rm rest} + \frac{1}{4\pi\alpha'}\left(\pi T_H + \sqrt{\pi^2 T_H^2 - 2\ell^2}\right)\right)\frac{(a+b)^2}{2}\sin(2\Theta_0) + ...,
$$

where the cut-off $r_m = 2\pi\alpha' m_{\rm rest}$.

The components of the drag force $a \neq b \neq 0$

$b=0$ The dependence of $\frac{dp_\theta}{dt}$ on the temperature is

$$\begin{aligned}
\frac{dp_\theta}{dt} &= 3\left(\ell^2 m_{\mathrm{rest}} + \frac{1}{4\pi\alpha'}\left(\pi T_H + \sqrt{\pi^2 T_H^2 - 2\ell^2}\right)\right)\frac{a^2}{2}\sin(2\Theta_0) + ... \quad , \\
\frac{dp_\phi}{dt} &= -\frac{1}{2\pi\alpha'}\mathfrak{p}\sin(\Theta_0)^2 a + \mathcal{O}(a^2), \quad \frac{dp_\psi}{dt} = 0.
\end{aligned}$$

Aref'eva, AG, Gourgoulhon'20

Supposing that the quark slowly moves with an angular velocity $(\omega_\theta, \omega_\phi, \omega_\psi)$.

$$\begin{aligned}
\frac{dp_\theta}{dt} &= -\frac{\omega_\theta r_H^2}{2\pi\alpha'} + 3\left(\ell^2 m_{\mathrm{rest}} + \frac{1}{4\pi\alpha'}\left(\pi T_H + \sqrt{\pi^2 T_H^2 - 2\ell^2}\right)\right)\frac{a^2}{2}\sin(2\Theta_0) + ..., \\
\frac{dp_\phi}{dt} &= -\frac{1}{2\pi\alpha'}(\omega_\phi r_H^2 - \mathfrak{p}\sin(\Theta_0)^2 a) + ..., \quad \frac{dp_\psi}{dt} = -\frac{1}{2\pi\alpha'}\omega_\psi r_H^2 + ...,
\end{aligned}$$

It is also instructive to obtain a relation to friction coefficients.

$$\mu = \left(\frac{\pi T}{\ell}\right)^2 \frac{1}{2m\pi\alpha'}, \quad p = \frac{m\omega}{\ell^2}.$$

Drag force $a \neq b \neq 0$ AG, Gourgoulhon, Usova'21

Non-rotating-at-infinity, a,b are arbitrary

The drag force components near the boundary of Kerr-AdS_5 $y \to +\infty$

$$\begin{aligned}
\frac{dp_\Theta}{dt} &= \Big(-\frac{2\tilde{C}_1}{\sin(2\Theta_0)} + \frac{y(a-b)}{(a+b)} + \frac{(a-b)}{(a+b)y} \\
&- \frac{\pi}{2}\frac{(a^2-b^2)(1-y_H^4) - a^2\mathfrak{p}^2 + b^2\mathfrak{q}^2}{\sqrt{y_H^2+1}(2y_H^2+1)(a+b)^2}\Big)\frac{(a+b)^2}{4\pi\alpha'}\sin(2\Theta_0) + \mathcal{O}((a+b)^4), \\
\frac{dp_\Phi}{dt} &= -\frac{\mathfrak{p}}{2\pi\alpha'}\sin^2(\Theta_0)a + \mathcal{O}(a^2), \quad \frac{dp_\Psi}{dt} = -\frac{\mathfrak{q}}{2\pi\alpha'}\cos^2(\Theta_0)b + \mathcal{O}(b^2).
\end{aligned}$$

As in the rotating-at-infinity frame we observe the divergent term with $y \to +\infty$, which can be associated to an infinite mass of the quark. Introducing a cut-off as $y_m = 2\pi\alpha' m_{\mathrm{rest}}$

$$\begin{aligned}
\frac{dp_\Theta}{dt} &\approx \frac{a-b}{a+b}\left(m_{\mathrm{rest}} + \frac{1}{4\pi\alpha'}\left(\pi T_H + \sqrt{\pi^2 T_H^2 - 2}\right)\right)\frac{(a+b)^2}{2}\sin(2\Theta_0) + \dots, \\
\frac{dp_\Phi}{dt} &= -\frac{1}{2\pi\alpha'}\mathfrak{p}\sin(\Theta_0)^2 a + \mathcal{O}(a^2), \quad \frac{dp_\Psi}{dt} = 0.
\end{aligned}$$

The form of the relations for the drag force in the rotating-at-infinity and static frames is similar up to a number coefficient.

CHECKED WITH SAGE

Wilson loop in Kerr-AdS_5 black hole

Kerr-AdS_5 black hole

$$\begin{aligned} ds^2 \quad &\simeq \quad -(1+y^2)dT^2 + \frac{dy^2}{1+y^2-\frac{2M}{\Delta^2 y^2}} + y^2(d\Theta^2 + \sin^2\Theta d\Phi^2 + \cos^2\Theta d\Psi^2) \\ &+ \quad \frac{2M}{\Delta^3 y^2}dT^2 + \frac{2Ma^2\sin^4\Theta}{\Delta^3 y^2}d\Phi^2 + \frac{2Mb^2\cos^4\Theta}{\Delta^3 y^2}d\Psi^2 - \\ &- \quad \frac{4Ma\sin^2\Theta}{\Delta^3 y^2}dTd\Phi - \frac{4Mb\cos^2\Theta}{\Delta^3 y^2}dTd\Psi + \frac{4Mab\sin^2\Theta\cos^2\Theta}{\Delta^3 y^2}d\Phi d\Psi, \end{aligned}$$

where $\Delta = 1 - a^2\ell^{-2}\sin^2\Theta - b^2\ell^{-2}\cos^2\Theta$.

String worldsheet parametrization

$$\tau = T, \qquad \sigma = \Phi, \qquad y = y(\Phi), \qquad \Phi \in [0, 2\pi L_\Phi].$$

The boundary conditions $y\left(-\frac{L_\Phi}{2}\right) = y\left(\frac{L_\Phi}{2}\right) = 0$.

Wilson loop in Kerr-AdS_5 AG&Tsegelnik'22

The Nambu-Goto action is

$$S_{NG} = \frac{T}{2\pi\alpha'} \int\limits_{-\frac{L_\Phi}{2}}^{\frac{L_\Phi}{2}} d\Phi \sqrt{y'^2 \frac{f_{\Delta^3}(y)}{f_{\Delta^2}(y)} + y^2 F_{\Delta^3}(y) \sin^2\Theta},$$

where we redefine

$$f_{\Delta^2}(y) \equiv 1 + y^2\ell^{-2} - \frac{2M}{\Delta^2 y^2}, \qquad f_{\Delta^3}(y) \equiv 1 + y^2\ell^{-2} - \frac{2M}{\Delta^3 y^2}$$

$$F_{\Delta^3}(y) = f_{\Delta^3}(y) + \frac{2Ma^2\sin^2\Theta}{y^4\Delta^3}\left(1 + y^2\ell^{-2}\right).$$

The integral of motion

$$\mathcal{H} = -\frac{y^2 F_{\Delta^3}(y)\sin^2\Theta}{\sqrt{y'^2 \frac{f_{\Delta^3}(y)}{f_{\Delta^2}(y)} + y^2 F_{\Delta^3}(y)\sin^2\Theta}}.$$

The turning point is defined by $y' = 0$, so

$$y \sin\Theta \sqrt{F_{\Delta^3}(y)}\Big|_{y=y_m} = \frac{1}{\ell C}, \quad y_m = y(\Phi_m)$$

The equation of motion is

$$y'^2 = y^2 F_{\Delta^3}(y)\left[C^2\ell^2 \sin^2\Theta y^2 F_{\Delta^3}(y) - 1\right]\frac{f_{\Delta^2}(y)}{f_{\Delta^3}(y)}\sin^2\Theta.$$

The renormalized NG action

$$S_{NG} = \frac{T}{\pi\alpha'}\left[\int_{y_m}^{\infty} dy\sqrt{\frac{f_{\Delta^3}(y)}{f_{\Delta^2}(y)}}\left(\frac{C\ell\sin\Theta y\sqrt{F_{\Delta^3}(y)}}{\sqrt{C^2\ell^2\sin^2\Theta y^2 F_{\Delta^3}(y) - 1}} - 1\right) - \int_{y_+}^{y_m} dy\sqrt{\frac{f_{\Delta^3}(y)}{f_{\Delta^2}(y)}}\right].$$

The distance between quarks L_Φ:

$$\frac{L_\Phi}{2} = \int_{y_m}^{\infty} dy\,\frac{1}{\sin\Theta y\sqrt{F_{\Delta^3}(y)}\sqrt{C^2\ell^2\sin^2\Theta y^2 F_{\Delta^3}(y) - 1}}\sqrt{\frac{f_{\Delta^3}(y)}{f_{\Delta^2}(y)}}.$$

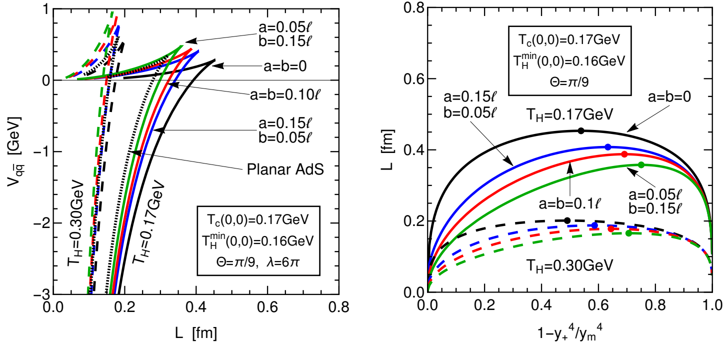


FIGURE: A) The behaviour of $V_{q\bar{q}}$ on the interquark distance L_Φ .B) L_Φ as a function of $1 - \frac{y_+^4}{y_m^4}$.

Outline

Summary

- Rotation has an influence on the first order phase transition:
  - it decreases the critical temperature;
  - the critical end point depends on some combination of the two rotation parameters.
- The drag force contains two terms; one of them is related to the slow motion of the quark with respect to the fluid, the second one is interpreted as a centripetal force.
  - The friction coefficient depends on the temperature quadratically;
  - the centripetal term depends on the temperature linearly;
  - the centripetal force vanishes in the case of two equal rotational parameters.
- In BL coordinates (rotating-at-infinity frame), the energy of the quark at rest in the zero temperature limit has a contribution from the rotational parameter, while in non-rotating-at-infinity frame (AdS coordinates) it does not.

Thank you for attention!