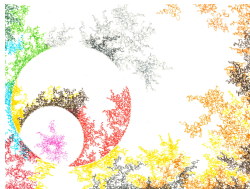


Black hole quasi-normal mode instability: (Missing degrees of freedom and asymptotic symmetries)

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Mathematical Aspects of Black Hole Theory
Observatoire de Paris, Meudon, 12-14 December 2022

- 1 The Problem in a nutshell: (asymptotically flat) BH QNM instability
- 2 Non-normal operators: spectral instability and Pseudospectrum
- 3 Hyperboloidal approach to QNMs: the Pöschl-Teller toy model
- 4 Black Hole QNM ultraviolet instability
- 5 Discussion, Conclusions and Perspectives: a low-regularity problem
 - Missing degrees of freedom and asymptotic symmetries

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BH quasi-normal modes as a spacetime probe

Black hole quasi-normal modes (QNMs): a cornerstone in Gravitation

Resonant response of black holes under linear perturbations: complex frequencies providing an invariant probe into the background spacetime geometry.

- i) Relativistic astrophysics and gravitational wave physics.
- ii) Gravity and the quantum: semiclassical gravity, AdS/CFT-fluid/gravity dualities, analogue gravity...
- iii) Mathematical relativity.
- iv) ... (Interdisciplinary physics)

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A first question:

Are QNMs “structurally stable” under small perturbations?

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A first question:

Are QNMs “structurally stable” under small perturbations?

A further (bold!) question:

Can we measure the ‘(effective) regularity’ of spacetime from the perturbation of quasi-normal mode (QNM) overtones?

Black Hole QNM instabilities [cf. Nollert 96, Nollert & Price 99]

Perturbation theory on a Schwarzschild Black Hole: spherically symmetric case

Scalar, electromagnetic and gravitational perturbations reduced to (Minkowski) 1+1 wave equation for $\phi_{\ell m}(t, r_*)$ with a potential V_ℓ [Regge-Wheeler 57, Zerilli 70]:

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V_\ell \right) \phi_{\ell m} = 0 \quad , \quad t \in]-\infty, \infty[\quad , \quad r_* \in]-\infty, \infty[$$

Schwarzschild quasi-normal modes

Convention for a “mode”: $\phi_{\ell m}(t, r_*) \sim e^{i\omega t} \hat{\phi}_{\ell m}(r_*)$.

“Spectral” problem with **“outgoing boundary”** conditions:

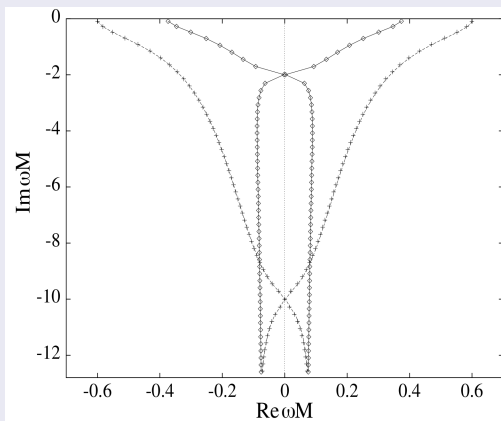
$$\left(-\frac{\partial^2}{\partial r_*^2} + V_\ell \right) \hat{\phi}_{\ell m} = \omega^2 \hat{\phi}_{\ell m} \quad , \quad r_* \in]-\infty, \infty[$$

$$\hat{\phi}_{\ell m} \sim e^{-i\omega r_*} \quad , \quad (r_* \rightarrow \infty) \quad , \quad \hat{\phi}_{\ell m} \sim e^{i\omega r_*} \quad , \quad (r_* \rightarrow -\infty)$$

Time evolution stability: $\text{Im}(\omega) > 0$. Exponential divergence of $\hat{\phi}_{\ell m}$ at $\pm\infty$.

Black Hole QNM instabilities [cf. Nollert 96, Nollert & Price 99]

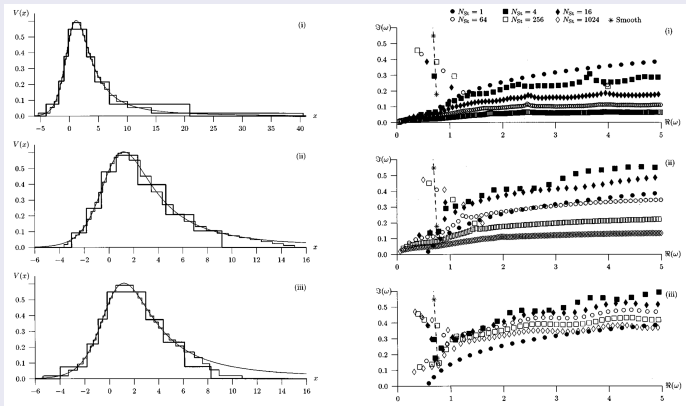
Schwarzschild gravitational QNMs



Schwarzschild QNMs ($\ell = 2$ diamonds, $\ell = 3$ crosses) [e.g. Kokkotas & Schmidt 99]
 QNM frequencies ω_n are invariant probes into the background geometry

Black Hole QNM instabilities [cf. Nollert 96, Nollert & Price 99]

Nollert's work on stair-case discretizations of Schwarzschild
(revisited in [Daghighi, Green & Morey 20, arXiv:2002.0725])



- Instability of the slowest decaying QNM (but ringdown “stability”).
- Instabilities of “highly damped QNMs”.

Posing the problem

Consider the operator on functions (defined on “appropriate” functional spaces) with non-compact-domain and with $V > 0$ with appropriate decay at infinity:

$$P_V = -\Delta + V$$

QNMs in the theory of Scattering Resonances

[Lax & Phillips, Vainberg; Sjöstrand, Zworski, Petkov, Iantchenko, ...many others; e.g. Dyatlov & Zworski 20]

Resolvent $R_V(\lambda) = (P_V - \lambda)^{-1}$ analytic $\text{Im}(\lambda) > 0$. Scattering resonances: poles of the meromorphic extension of (truncated) $R_V(\lambda)$ to $\text{Im}(\lambda) < 0$.

QNMs as a “proper” eigenvalue problem: **non-selfadjoint operators**

- “Complex scaling” [Simon 78, Reed & Simon 78, Sjöstrand...]: not the approach followed here.
- Hyperboloidal approach [Friedman & Schutz 75, Schmidt 93, Bizon, Zenginoglu 11, Vasy 13, Warnick 15, Ansorg & P.-Macedo 16, Gajic & Warnick 19, Bizon et al. 20, Galkowski & Zworski 20, ...]

Problem in terms of “eigenvalue problem” of **non-selfadjoint operator** L :

$$L u_{\ell m} = \omega u_{\ell m} \quad , \quad u_{\ell m} \in H \text{ (Hilbert space)}$$

- **Geometric boundary conditions:** Null infinity reached by hyperboloidal slices.
- **Regularity conditions on $u_{\ell m}$:** choice of appropriate H , then $\omega \in \sigma_p(L)$.

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Spectral Theorem. Normal and 'non-normal' operators

Normal operators: Spectral Theorem

- **Normality:** denoting the adjoint matrix by L^\dagger , then L is normal iff

$$[L, L^\dagger] = LL^\dagger - L^\dagger L = 0$$

Matrix examples: symmetric, hermitian, orthogonal, unitary...

- **Spectral Theorem** ("moral statement"):
 L is normal iff is unitarily diagonalisable.

Note: this depends on the adjoint L^\dagger , then on the Hilbert space (scalar product).

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'Non-normal' operators, $[L, L^\dagger] \neq 0$: no Spectral Theorem

- Completeness more difficult to study.
- Eigenvectors not necessarily orthogonal.
- **Spectral instabilities.**

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Methodology here: "exploration" stage

Numerical spectral methods: Chebyshev polynomials truncations.

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Example of spectral instability

$$L = a \frac{d^2}{dx^2} + b \frac{d}{dx} + c, \quad a, b, c \in \mathbb{R}$$

acting on functions in $L^2([0, 1])$, with homogeneous Dirichlet conditions (Chebyshev finite-dimensional matrix approximates).

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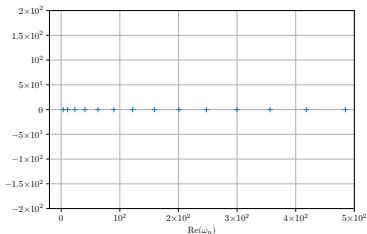
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Eigenvalues of L with $n = N + 1 = 51$ points

$$a = -1, b = 0, c = 1, \epsilon = 0$$

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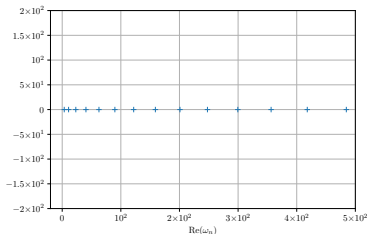
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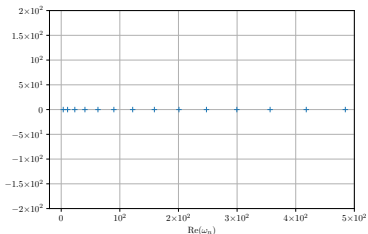
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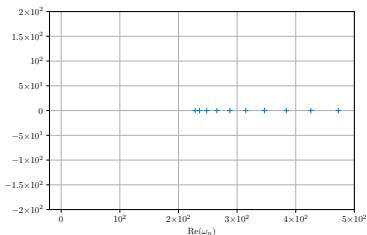
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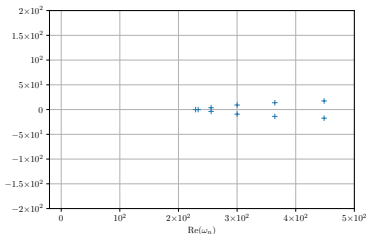
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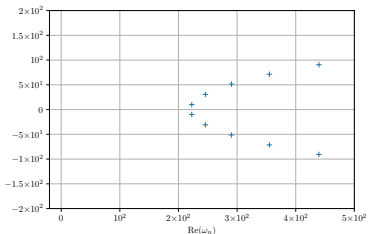
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$$a = -1, \quad b = 30, \quad c = 1, \quad \epsilon = 10^{-8}$$

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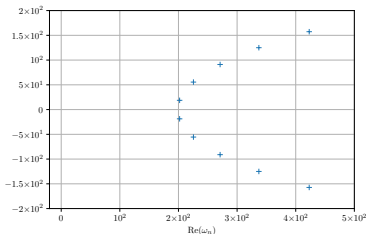
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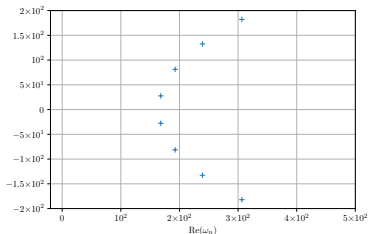
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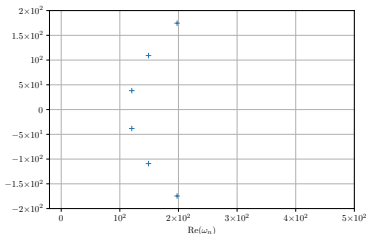
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Spectral (in)stability: eigenvalue condition number

Left- and right-eigenvectors, respectively u_i and v_i , of A

$$A^\dagger u_i = \bar{\lambda}_i u_i \quad (\Leftrightarrow u_i^\dagger A = \lambda_i u_i^\dagger) \quad , \quad Av_i = \lambda_i v_i \quad , \quad i \in \{1, \dots, n\} \quad ,$$

Perturbation theory of eigenvalues [cf. Kato 80, ...; e.g. Trefethen, Embree 05]:

$$A(\epsilon) = A + \epsilon \delta A \quad , \quad \|\delta A\| = 1 .$$

$$|\lambda_i(\epsilon) - \lambda_i| = \epsilon \frac{|\langle u_i, \delta A v_i(\epsilon) \rangle|}{|\langle u_i, v_i \rangle|} \leq \epsilon \frac{\|u_i\| \|\delta A v_i\|}{|\langle u_i, v_i \rangle|} + O(\epsilon^2) \leq \epsilon \frac{\|u_i\| \|v_i\|}{|\langle u_i, v_i \rangle|} + O(\epsilon^2) .$$

Eigenvalue condition number: $\kappa(\lambda_i)$

$$\kappa(\lambda_i) = \frac{\|u_i\| \|v_i\|}{|\langle u_i, v_i \rangle|}$$

Spectral (in)stability and Pseudospectrum

Pseudospectrum

Given $\epsilon > 0$, the ϵ -pseudospectrum $\sigma_\epsilon(L)$ of L is defined as [e.g. Trefethen & Embree 05]:

$$\begin{aligned} \sigma_\epsilon(L) &= \{\lambda \in \mathbb{C}, \text{ such that } \lambda \in \sigma(L + \delta L) \text{ for some } \delta L \text{ with } \|\delta L\| < \epsilon\} \\ &= \{\lambda \in \mathbb{C}, \text{ such that } \|Lv - \lambda v\| < \epsilon \text{ for some } v \text{ with } \|v\| = 1\} \\ &= \{\lambda \in \mathbb{C}, \text{ such that } \|(\lambda I - L)^{-1}\| > \epsilon^{-1}\} \end{aligned}$$

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Normal case: bounds on the norm of the resolvent $R_L(\lambda) = (\lambda I - L)^{-1}$

Given $\lambda \in \mathbb{C}$ and $\sigma(L)$ the spectrum of L , it holds

$$\|(\lambda I - L)^{-1}\|_2 = \frac{1}{\text{dist}(\lambda, \sigma(L))}$$

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Non-normal case: bad control on the resolvent $R_L(\lambda)$. Pseudospectrum

The norm of the resolvent can become very large far from the spectrum:

$$\|(\lambda I - L)^{-1}\|_2 \leq \frac{\kappa}{\text{dist}(\lambda, \sigma(L))}$$

where κ is a “condition number” assessing the lack of proportionality of ‘left’ and ‘right’ eigenvectors of L , and can become very large in the non-normal case.

Bauer-Fike theorem. Random perturbations

Pseudospectrum and condition number: Bauer-Fike theorem [e.g Trefethen & Embree 05]

Defining “tubular neighbourhood” of radius ϵ around $\sigma(A)$

$$\Delta_\epsilon(A) = \{\lambda \in \mathbb{C} : \text{dist}(\lambda, \sigma(A)) < \epsilon\},$$

it holds: $\Delta_\epsilon(A) \subseteq \sigma_\epsilon(A)$. For normal operators: $\sigma_\epsilon(A) = \Delta_\epsilon(A)$.

Non-normal case, $\kappa(\lambda_i) \neq 1$, it holds (for small ϵ):

$$\sigma_\epsilon(A) \subseteq \bigcup_{\lambda_i \in \sigma(A)} \Delta_{\epsilon\kappa(\lambda_i) + O(\epsilon^2)}(\{\lambda_i\}),$$

Therefore $\sigma_\epsilon(A)$ larger tubular neighbourhood of radius $\sim \epsilon\kappa(\lambda_i)$.

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Random perturbations and Pseudospectrum

Random perturbations ΔL with $\|\delta L\| < \epsilon$ “push” eigenvalues into $\sigma_\epsilon(A)$, providing an insightful and systematic manner of exploring $\sigma_\epsilon(L)$.

Bauer-Fike theorem. Random perturbations

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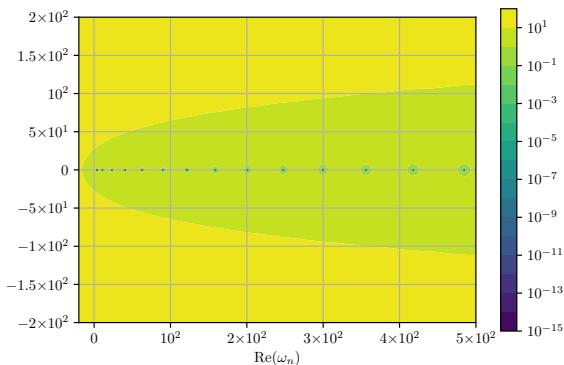
The 'role' of random perturbations [Sjöstrand 19; Hager 05, Montrieux, Nonnenmacher, Vogel,...]

Random perturbations improve the analytical behaviour of $R_L(\lambda)$!!!

Spectral (in)stability and Pseudospectrum: illustration

Pseudospectrum of: $L = a \frac{d^2}{dx^2} + b \frac{d}{dx} + c + \epsilon E_{\text{Random}}$

Spectrum and Pseudospectrum of L with $\log\|\text{Random}\|_2 = -50$

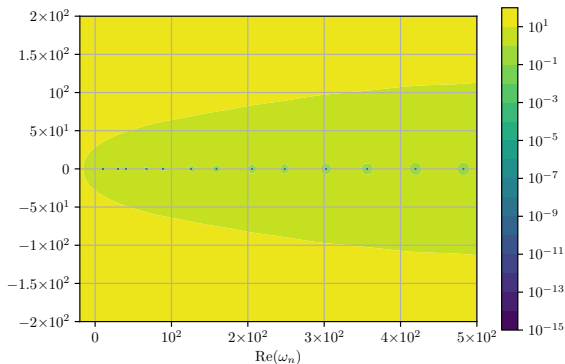


$$a = -1, b = 0, c = 1, \epsilon = 0$$

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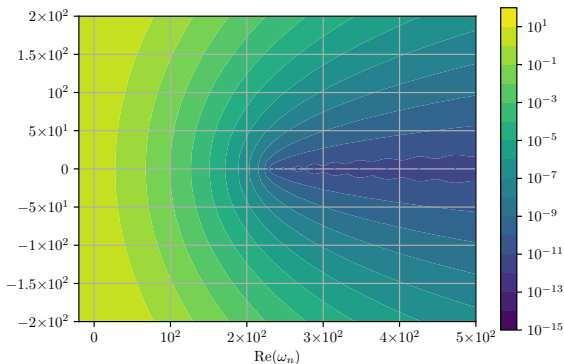


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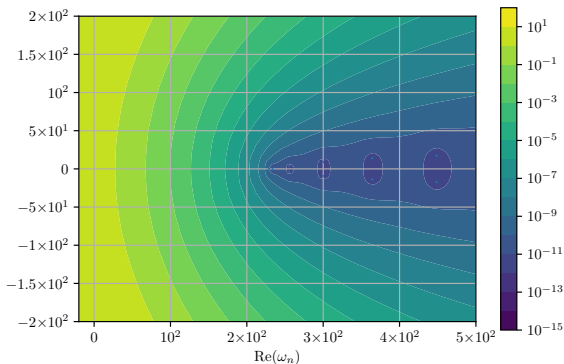


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Pseudospectrum of: $L = a \frac{d^2}{dx^2} + b \frac{d}{dx} + c + \epsilon E_{\text{Random}}$

Spectrum and Pseudospectrum of L with $\log\|\text{Random}\|_2 = -10$

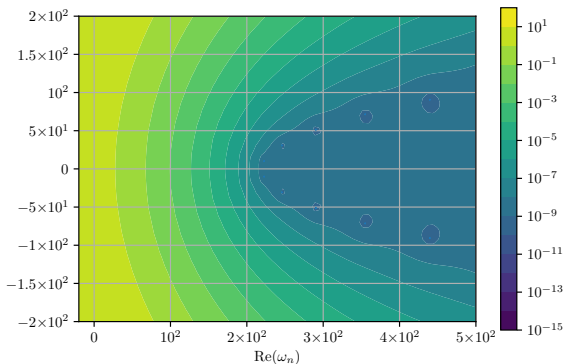


$$a = -1, b = 30, c = 1, \epsilon = 10^{-10}$$

Spectral (in)stability and Pseudospectrum: illustration

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Spectrum and Pseudospectrum of L with $\log\|\text{Random}\|_2 = -8$

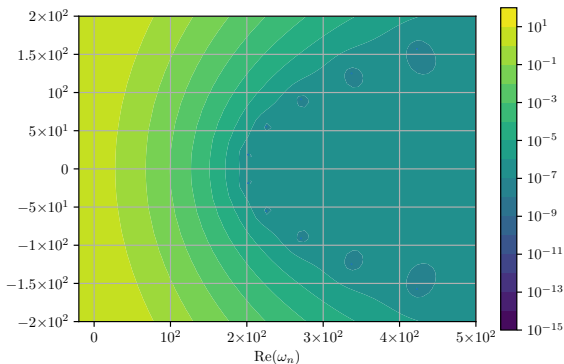


$$a = -1, b = 30, c = 1, \epsilon = 10^{-8}$$

Spectral (in)stability and Pseudospectrum: illustration

Pseudospectrum of: $L = a \frac{d^2}{dx^2} + b \frac{d}{dx} + c + \epsilon E_{\text{Random}}$

Spectrum and Pseudospectrum of L with $\log\|\text{Random}\|_2 = -6$

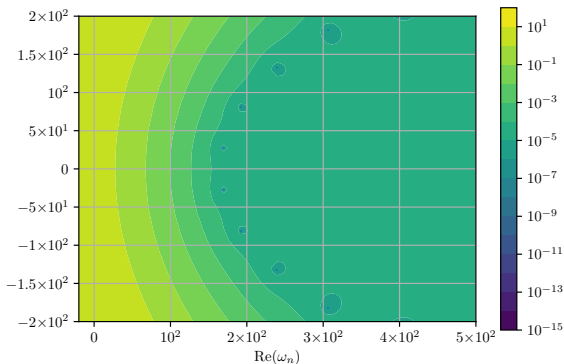


$$a = -1, b = 30, c = 1, \epsilon = 10^{-6}$$

Spectral (in)stability and Pseudospectrum: illustration

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Spectrum and Pseudospectrum of L with $\log\|\text{Random}\|_2 = -4$

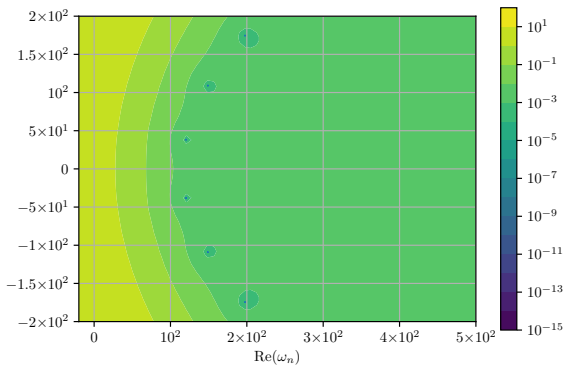


$$a = -1, b = 30, c = 1, \epsilon = 10^{-4}$$

Spectral (in)stability and Pseudospectrum: illustration

Pseudospectrum of: $L = a \frac{d^2}{dx^2} + b \frac{d}{dx} + c + \epsilon E_{\text{Random}}$

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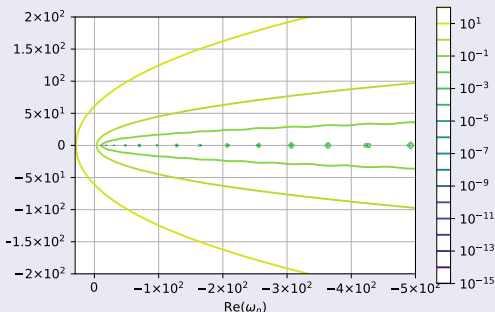
The relevance of the scalar product: assessing large/small

The illustrative operator: $L = a \frac{d^2}{dx^2} + b \frac{d}{dx} + c$, $a, b, c \in \mathbb{R}$

- Non-selfadjoint in standard $L^2([0, 1])$ for $b \neq 0$.
- Formally normal!
- Non-normal: domain of $L^\dagger L$ and LL^\dagger different.
- But actually self-adjoint...

Cast in Sturm-Liouville form: selfadjoint for appropriate scalar product $\langle \cdot, \cdot \rangle_w$!!!

Pseudospectrum using the L^2 -inner-product



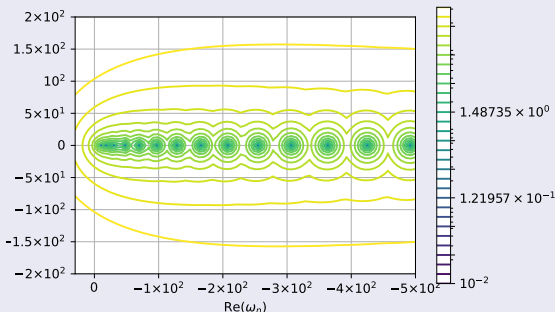
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Pseudospectrum using Gram Matrix = SturmLiouville-w



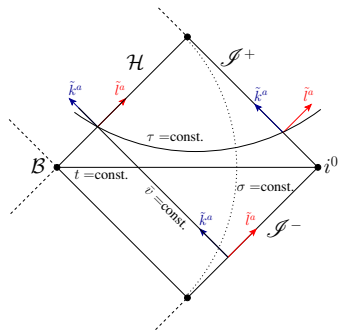
Scheme

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Hyperboloidal slices: geometric outgoing BCs at \mathcal{I}^+

Hyperboloidal approach to QNMs

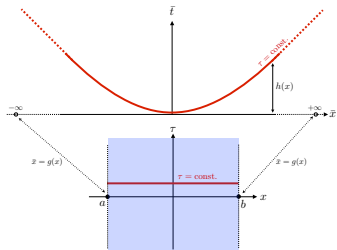
- **Spectral problem:** homogeneous wave equation with purely outgoing boundary conditions.
- Outgoing BCs naturally imposed at \mathcal{I}^+ .
- Outgoing BCs actually “incorporated” at \mathcal{I}^+ :
 - Geometrically: null cones outgoing.
 - Analytically: BCs encoded into a singular operator, “**BCs as regularity conditions**”.
- **Eigenfunctions** do not diverge when $x \rightarrow \infty$: actually **integrable**. Key to Hilbert space.



Hyperboloidal slices: geometric outgoing BCs at \mathcal{I}^+

Hyperboloidal approach to QNMs

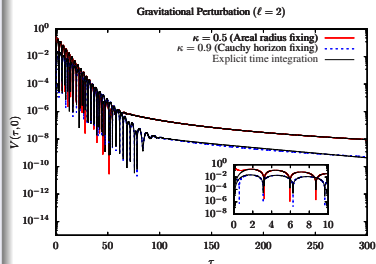
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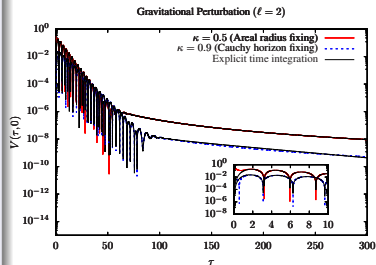
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1. Wave problem in spherically symmetric asymptotically flat case

As starting point, consider the problem for a $\phi_{\ell m}$ mode in tortoise coordinates:

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V_\ell \right) \phi_{\ell m} = 0 \quad , \quad t \in]-\infty, \infty[\quad , \quad r_* \in]-\infty, \infty[$$

Compactification along hyperboloidal slices

2. Choice of hyperboloidal foliation and compactification

Make the change to Bizoń-Mach variables [Bizoń & Mach 17]:

$$\begin{cases} \tau &= t - \ln(\cosh r_*) \\ x &= \tanh r_* \end{cases}, \quad \tau \in]-\infty, \infty[, \quad x \in]-1, 1[$$

- 1 $\tau = \text{const.}$ defines a hyperboloidal slicing.
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3. Wave equation in hyperboloidal coordinates: no boundary conditions allowed

For $x = \pm 1$, $V_\ell = 0$. In the interior, $x \in]-1, 1[$:

$$\left(\partial_\tau^2 + 2x\partial_\tau\partial_x + \partial_\tau + 2x\partial_x - (1-x^2)\partial_x^2 + \tilde{V}_\ell \right) \phi_{\ell m} = 0,$$

with $\tilde{V}_\ell = \frac{V_\ell}{(1-x^2)}$.

Wave equation: reduction to first order system

4. Evolution equation in first order form

Introducing the auxiliary field

$$\psi_{\ell m} = \partial_\tau \phi_{\ell m} ,$$

we can write the wave equation in first-order form:

$$\partial_\tau \begin{pmatrix} \phi_{\ell m} \\ \psi_{\ell m} \end{pmatrix} = \left(\begin{array}{c|c} 0 & 1 \\ \hline (1-x^2)\partial_x^2 - 2x\partial_x - \tilde{V}_{\ell m} & -(2x\partial_y + 1) \end{array} \right) \begin{pmatrix} \phi_{\ell m} \\ \psi_{\ell m} \end{pmatrix} .$$

Spectral problem: first order formulation

5. Eigenvalue problem for a non-selfadjoint operator, no BCs

Our problem: study

$$L \begin{pmatrix} \phi_{\ell m} \\ \psi_{\ell m} \end{pmatrix} = \omega \begin{pmatrix} \phi_{\ell m} \\ \psi_{\ell m} \end{pmatrix}, \quad L = \left(\begin{array}{c|c} 0 & 1 \\ L_1 & L_2 \end{array} \right),$$

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$$L_1 = (1 - x^2)\partial_x^2 - 2x\partial_x - \tilde{V}_{\ell m}, \quad L_2 = -(2x\partial_x + 1).$$

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QNM problem as a “proper” eigenvalue problem. But... **Hilbert space?**Spectral problem in a Hilbert space with **“Energy” scalar product** ($\tilde{V} > 0$):

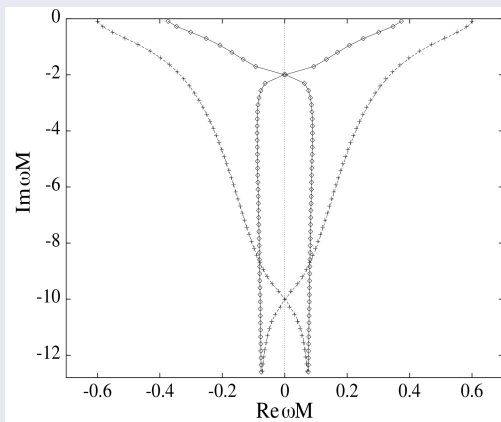
$$\left\langle \begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix}, \begin{pmatrix} \phi_2 \\ \psi_2 \end{pmatrix} \right\rangle_E = \int_{\Sigma_\tau} \left(\bar{\psi}_1 \psi_2 + (1-x^2) \partial_x \bar{\phi}_1 \partial_x \phi_2 + \tilde{V} \bar{\phi}_1 \phi_2 \right) d\Sigma_\tau$$

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Black Hole QNM instabilities [cf. Nollert 96, Nollert & Price 99]

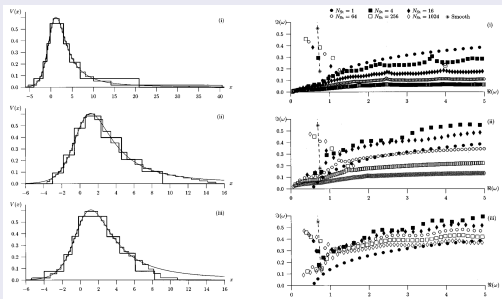
Schwarzschild gravitational QNMs



Schwarzschild QNMs ($\ell = 2$ diamonds, $\ell = 3$ crosses) [e.g. Kokkotas & Schmidt 99]

Black Hole QNM instabilities [cf. Nollert 96, Nollert & Price 99]

Nollert's work on stair-case discretizations of Schwarzschild
(revisited in [Daghighi, Green & Morey 20, arXiv:2002.0725])



- Instability of the slowest decaying QNM (but ringdown “stability”).
- Instabilities of “highly damped QNMs”.
- Various interests in BH QNM perturbations:
 - i) “Dirty” astrophysical black holes [Leung et al. 97; Barausse, Cardoso & Pani 14;...]
 - ii) Quantum (highly damped QNMs/high frequency instability) [Hod 98; Maggiore 08; Babb, Daghighi & Kunstatter 11; Ciric, Konjik & Samsarov 19, Olmedo; ...].

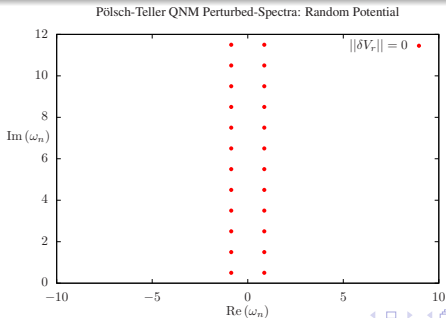
Test-bed study: Pöschl-Teller potential

Pöschl-Teller potential [JLJ, Macedo & Al Sheikh 21] (toy-model in [Bizoń, Chmaj & Mach 20])

$$V = V_o \operatorname{sech}^2(r_*) = V_o(1 - y^2) \implies \boxed{\tilde{V} = V_o}$$

Particularly simple form (scalar field in de Sitter, $m^2 = V_o$ [Bizoń, Chmaj & Mach 20])

- Integrable potential (QNM completeness [Beyer 99] with $m^2 = V_o!$).
- QNM frequencies: $\omega_n^\pm = -is_n^\pm = \pm \frac{\sqrt{3}}{2} + i(n + \frac{1}{2})$
- Here, eigenfunctions are Jacobi polynomials: $\phi_n(y) = P_n^{(s_n^\pm, s_n^\pm)}(y)$.



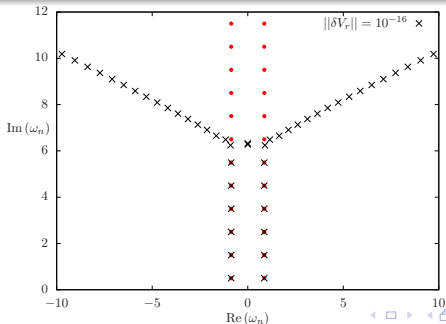
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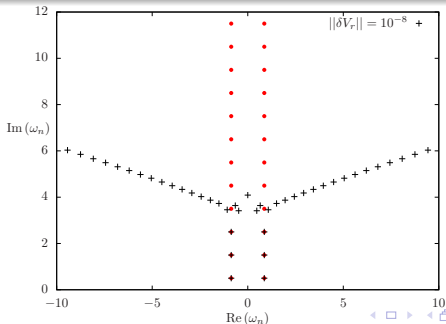
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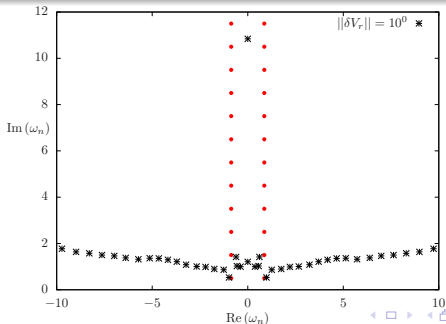
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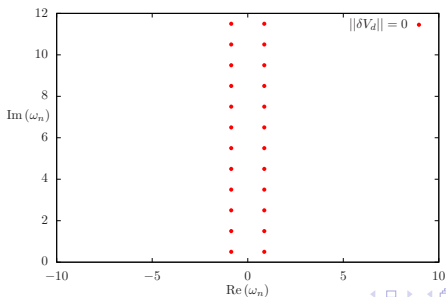
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Pöschl-Teller QNM Perturbed-Spectra: Deterministic Potential



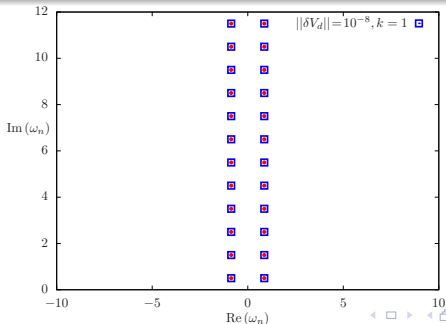
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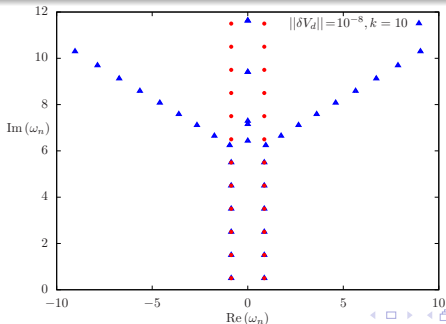
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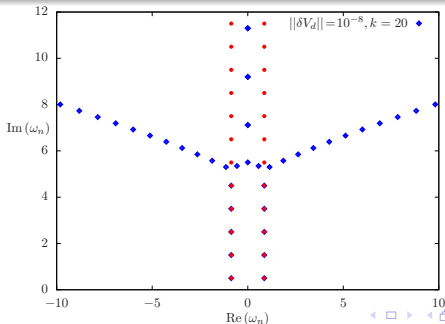
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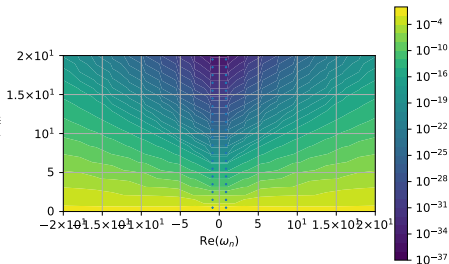
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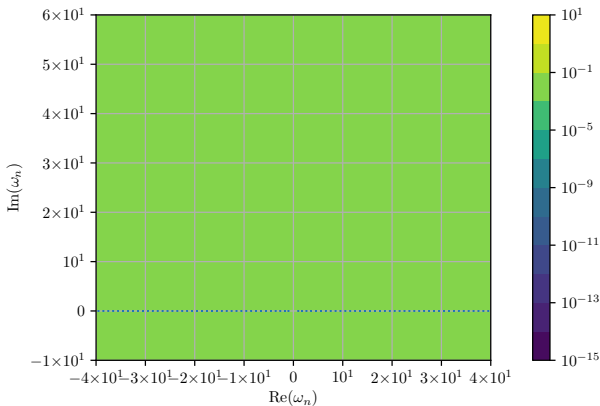
Poschel-teller Spectrum and Pseudospectrum of L



Test-bed study: Pöschl-Teller potential

Consistency check: self-adjoint case $\hat{L}_2 = 0$.

Spectrum and Pseudospectrum of L with $\log\|\text{Random}\|_2 = -50$



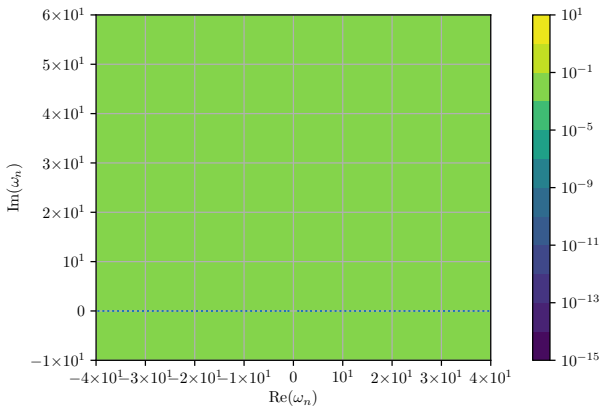
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$$\epsilon = 0$$

Test-bed study: Pöschl-Teller potential

Consistency check: self-adjoint case $\hat{L}_2 = 0$.

Spectrum and Pseudospectrum of L with $\log\|\text{Random}\|_2 = -5$



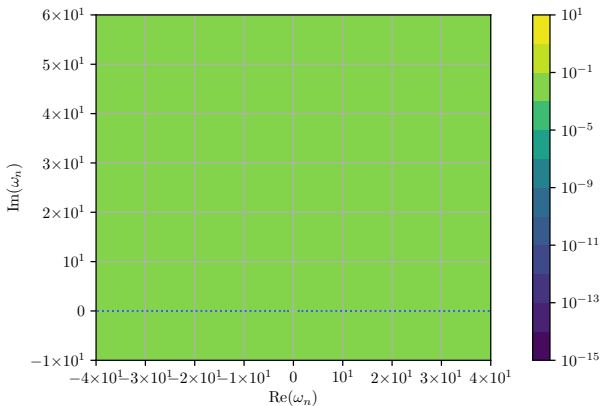
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$$\epsilon = 10^{-5}$$

Test-bed study: Pöschl-Teller potential

Consistency check: self-adjoint case $\hat{L}_2 = 0$.

Spectrum and Pseudospectrum of L with $\log\|\text{Random}\|_2 = -1$



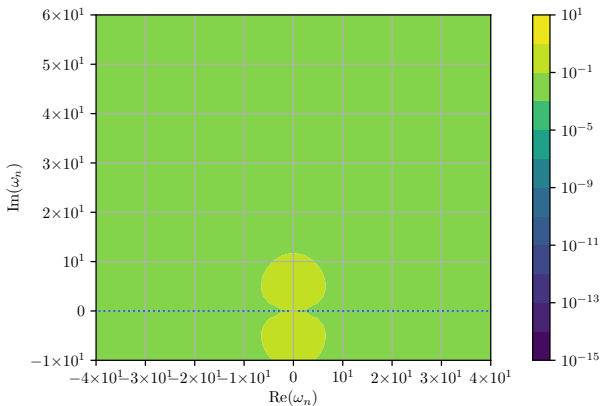
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$$\epsilon = 10^{-1}$$

Test-bed study: Pöschl-Teller potential

Consistency check: self-adjoint case $\hat{L}_2 = 0$.

Spectrum and Pseudospectrum of L with $\log\|\text{Random}\|_2 = 0$



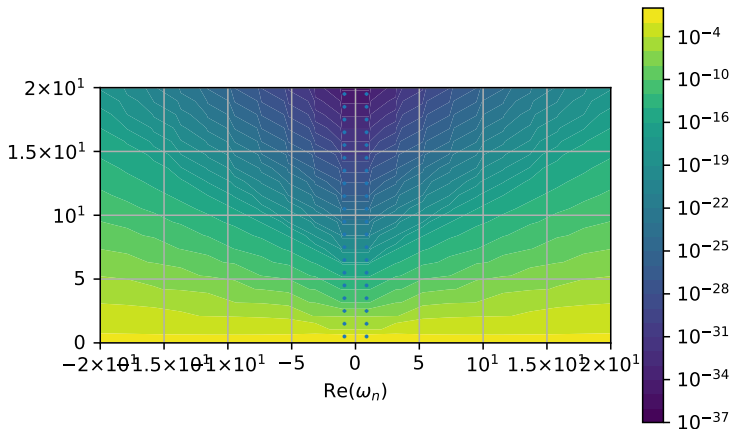
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$$\epsilon = 10^0$$

Test-bed study: Pöschl-Teller potential

QNM problem: $\hat{L}_2 \neq 0$.

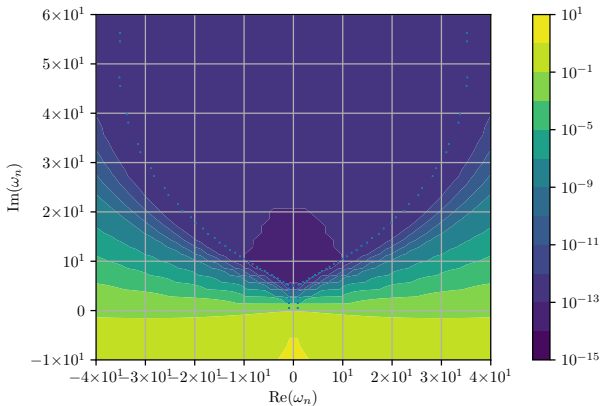
Poschel-teller Spectrum and Pseudospectrum of L



Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

Spectrum and Pseudospectrum of L with $\log\|\text{Random}\|_2 = -15$



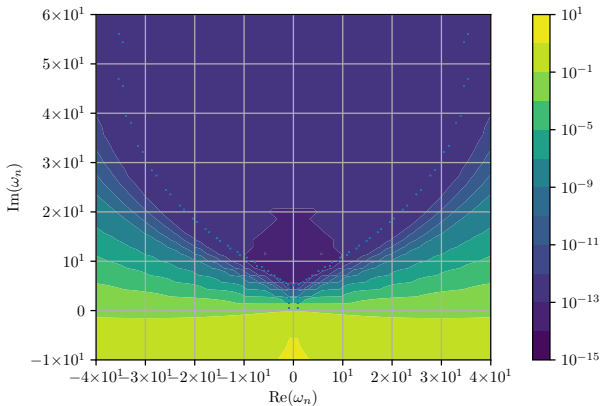
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$$\epsilon = 10^{-15}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

Spectrum and Pseudospectrum of L with $\log\|\text{Random}\|_2 = -14$



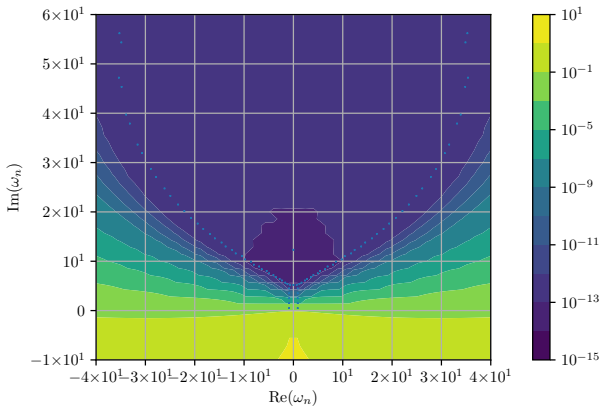
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$$\epsilon = 10^{-14}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

Spectrum and Pseudospectrum of L with $\log\|\text{Random}\|_2 = -13$



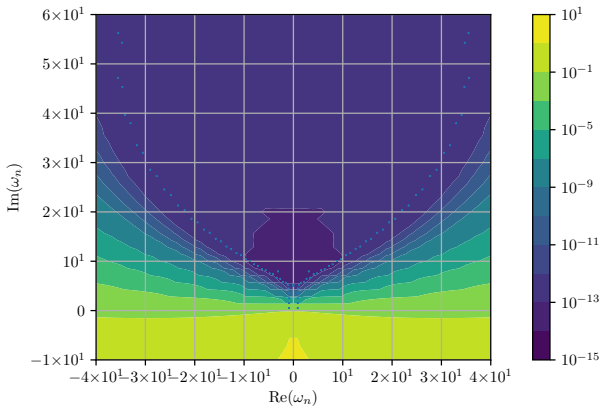
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$$\epsilon = 10^{-13}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

Spectrum and Pseudospectrum of L with $\log\|\text{Random}\|_2 = -12$



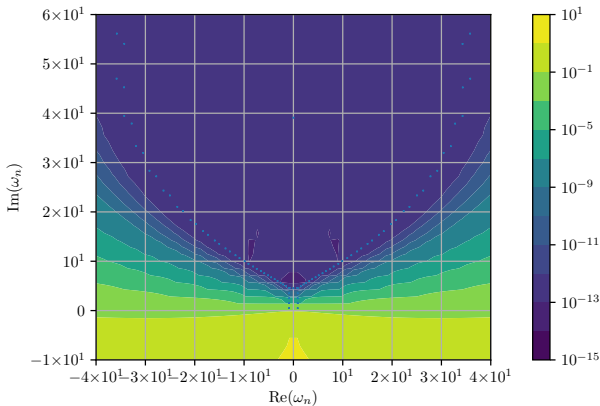
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$$\epsilon = 10^{-12}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

Spectrum and Pseudospectrum of L with $\log\|\text{Random}\|_2 = -11$



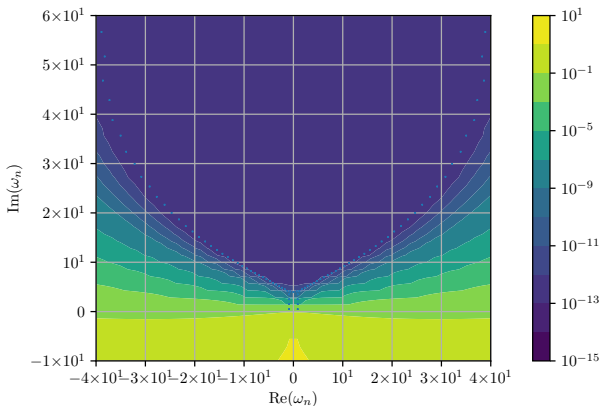
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$$\epsilon = 10^{-11}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

Spectrum and Pseudospectrum of L with $\log\|\text{Random}\|_2 = -10$



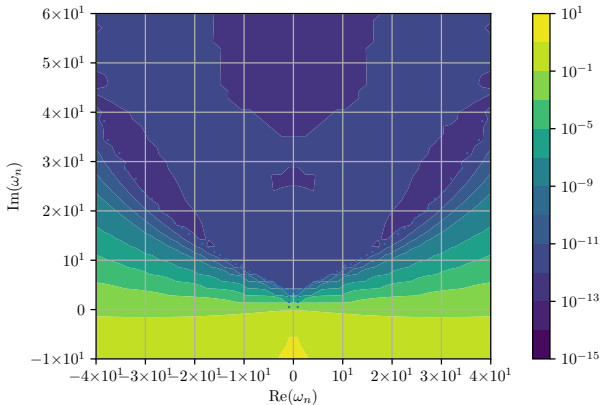
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$$\epsilon = 10^{-10}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

Spectrum and Pseudospectrum of L with $\log\|\text{Random}\|_2 = -9$



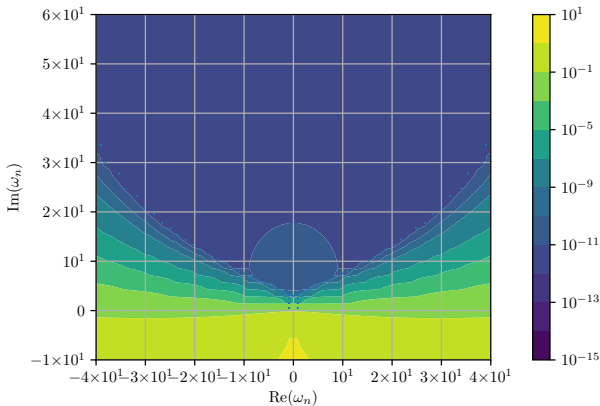
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$$\epsilon = 10^{-9}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

Spectrum and Pseudospectrum of L with $\log\|Random\|_2 = -8$



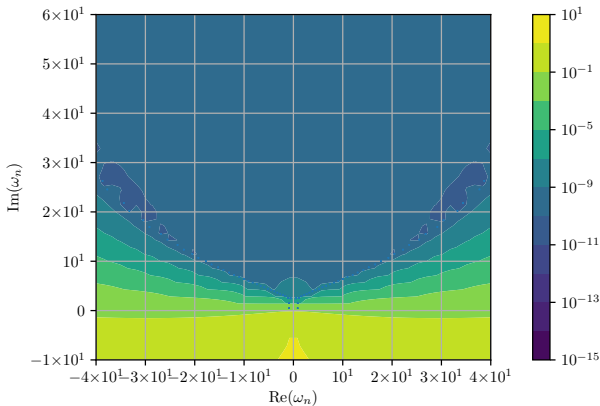
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$$\epsilon = 10^{-8}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

Spectrum and Pseudospectrum of L with $\log\|Random\|_2 = -7$



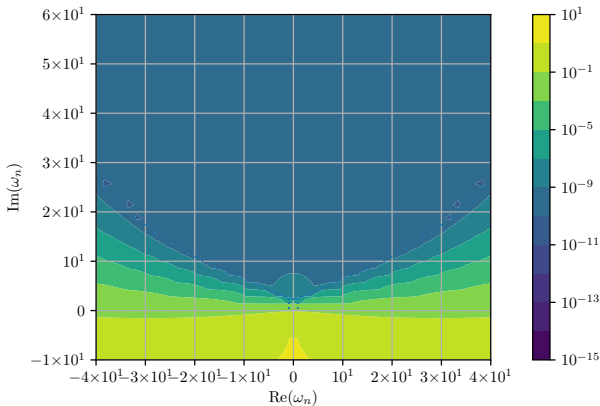
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$$\epsilon = 10^{-7}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

Spectrum and Pseudospectrum of L with $\log\|Random\|_2 = -6$



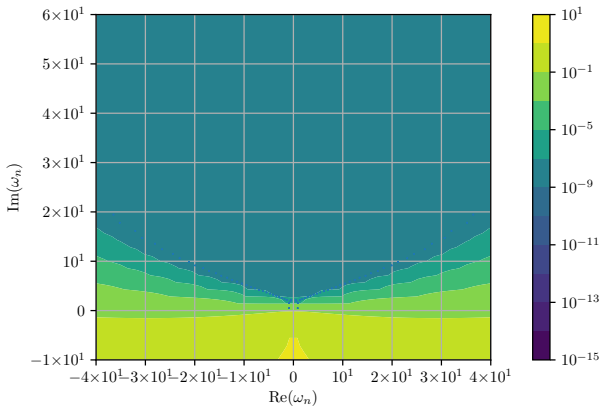
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$$\epsilon = 10^{-6}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

Spectrum and Pseudospectrum of L with $\log\|Random\|_2 = -5$



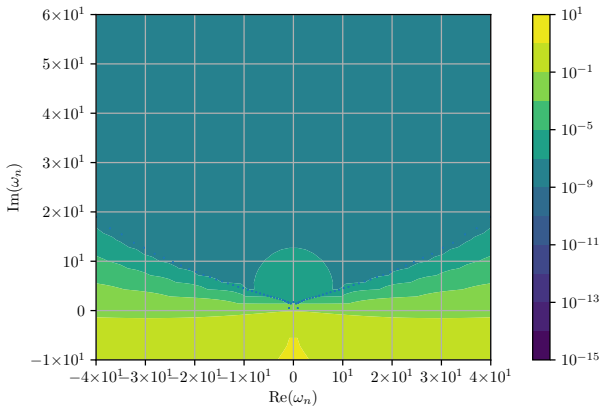
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$$\epsilon = 10^{-5}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

Spectrum and Pseudospectrum of L with $\log\|Random\|_2 = -4$



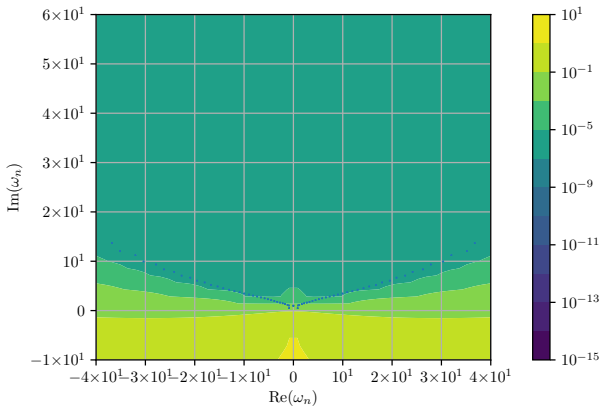
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$$\epsilon = 10^{-4}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

Spectrum and Pseudospectrum of L with $\log\|Random\|_2 = -3$



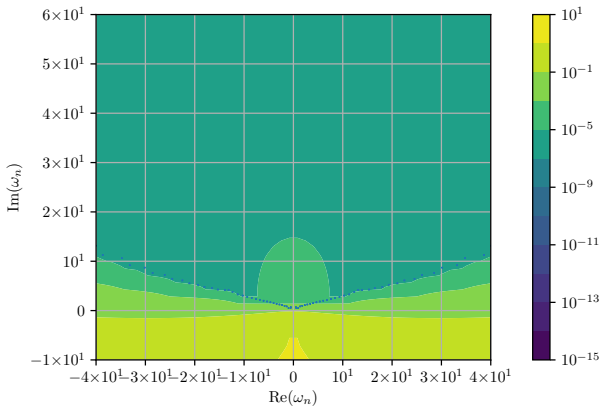
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$$\epsilon = 10^{-3}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

Spectrum and Pseudospectrum of L with $\log\|\text{Random}\|_2 = -2$



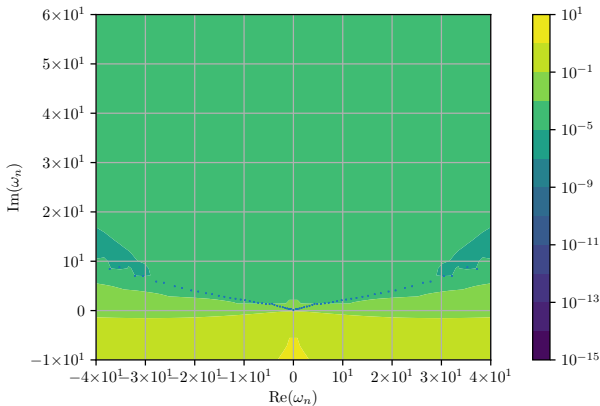
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$$\epsilon = 10^{-2}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

Spectrum and Pseudospectrum of L with $\log\|Random\|_2 = -1$



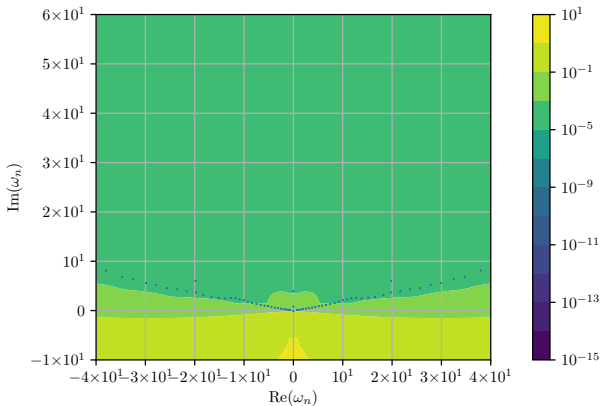
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$$\epsilon = 10^{-1}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

Spectrum and Pseudospectrum of L with $\log\|\text{Random}\|_2 = 0$



$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$$\epsilon = 10^0$$

Test-bed study: Pöschl-Teller potential

From this we learn:

- **ALL overtones QNMs, unstable under high frequency perturbations:** instability grows as damping grows [JLJ, Macedo & Al Sheikh 21].
- Perturbations make **QNMs to “migrate” towards Pseudospectrum contour lines** (“extended pattern”, cf. Bauer-Fike theorem).
- **Slowest damped QNM, stable under high frequency perturbations:**
 - Directly from the Pseudospectrum.
 - From the size of the needed perturbations.
- It can be repeated with deterministic **high frequency k** perturbations.
- For low frequency perturbations: much milder effect.

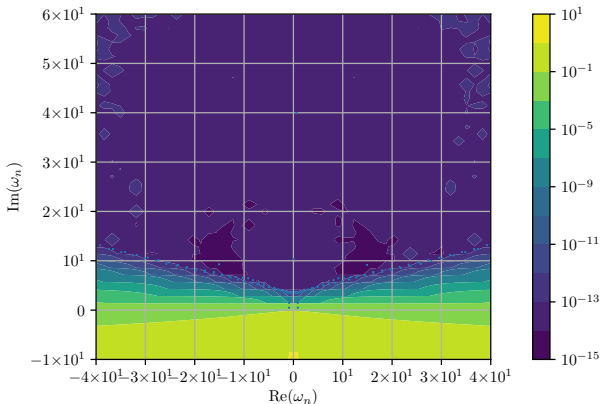
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon \cos(2\pi ky) \quad , \quad k \gg 1$$

- It can also be seen:
Slowest damped QNM unstable under “infrared perturbations” (cutting V at large distances): **Nollert’s instability of the fundamental QNM.**

Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution N . Fixed random perturbation $\epsilon = 10^{-16}$.

Spectrum and Pseudospectrum of L with $\log\|\text{Random}\|_2 = -50$



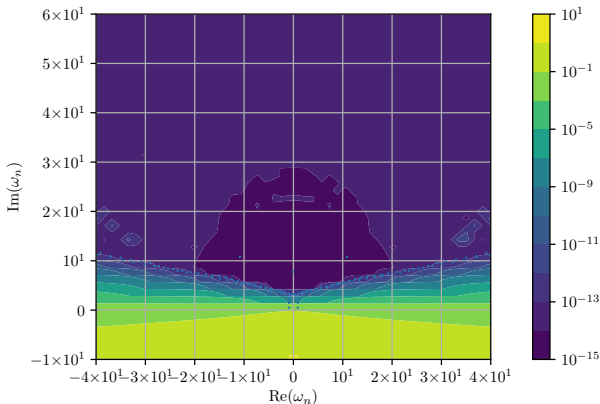
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$$N = 120$$

Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution N . Fixed random perturbation $\epsilon = 10^{-16}$.

Spectrum and Pseudospectrum of L with $\log\|\text{Random}\|_2 = -50$



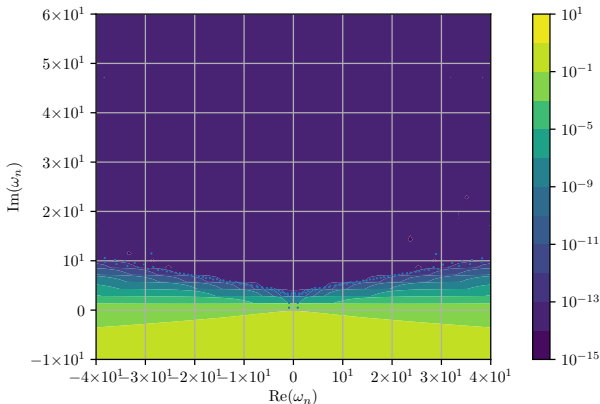
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$$N = 140$$

Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution N . Fixed random perturbation $\epsilon = 10^{-16}$.

Spectrum and Pseudospectrum of L with $\log\|\text{Random}\|_2 = -50$



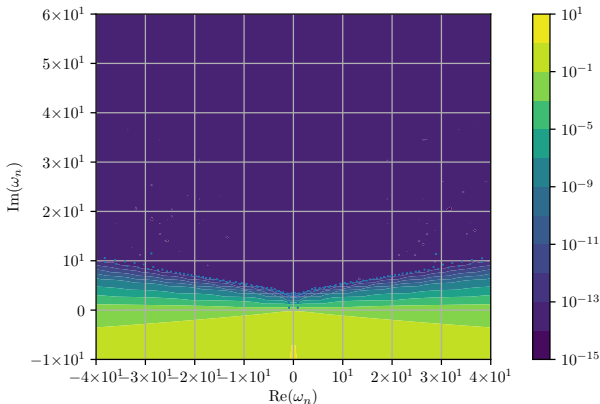
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$$N = 160$$

Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution N . Fixed random perturbation $\epsilon = 10^{-16}$.

Spectrum and Pseudospectrum of L with $\log\|\text{Random}\|_2 = -50$



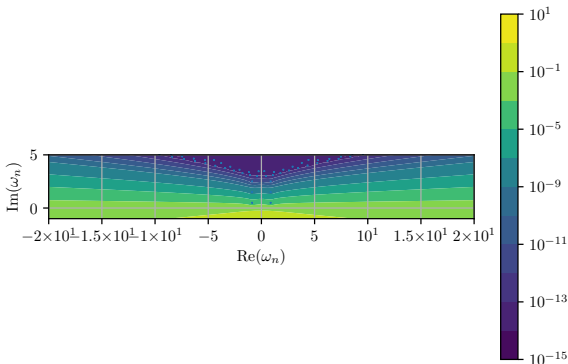
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$$N = 160$$

Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution N . Fixed random perturbation $\epsilon = 10^{-16}$.

Spectrum and Pseudospectrum of L with $\log\|\text{Random}\|_2 = -50$



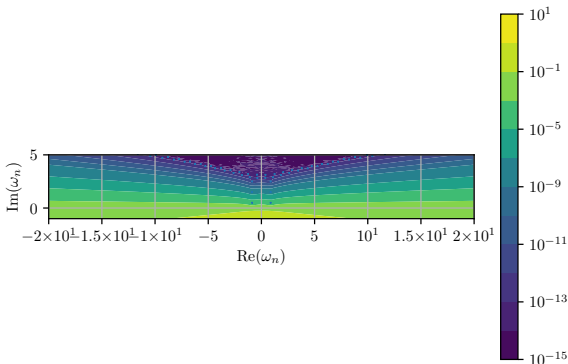
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$$N = 160$$

Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution N . Fixed random perturbation $\epsilon = 10^{-16}$.

Spectrum and Pseudospectrum of L with $\log\|\text{Random}\|_2 = -50$



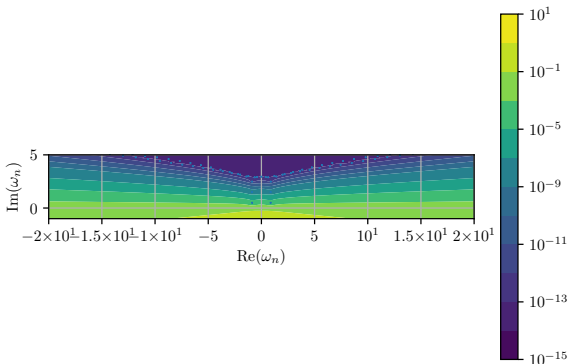
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$$N = 180$$

Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution N . Fixed random perturbation $\epsilon = 10^{-16}$.

Spectrum and Pseudospectrum of L with $\log\|\text{Random}\|_2 = -50$



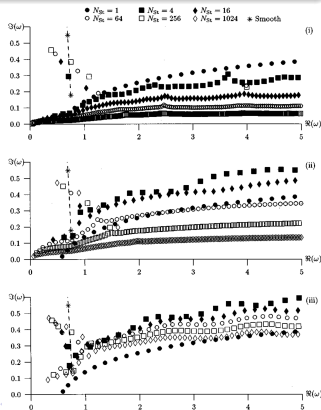
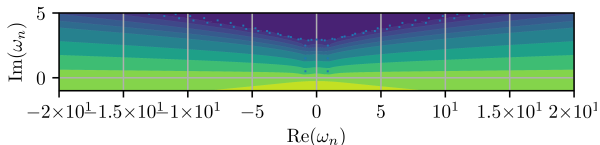
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$$N = 200$$

Test-bed study: Pöschl-Teller potential

From this we learn:

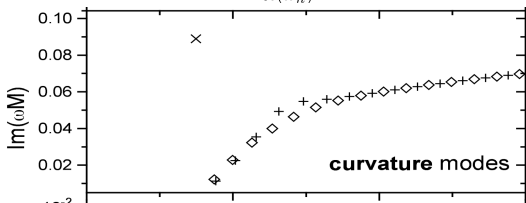
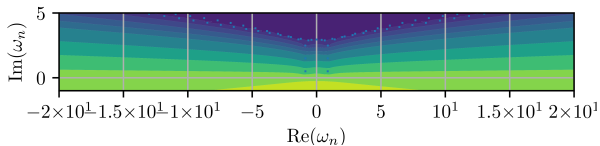
- Pseudospectrum contour lines seem to converge to QNM curves observed by Nollert [cf. Nollert 96].
- Similarity to Neutron-Star “w-modes”.



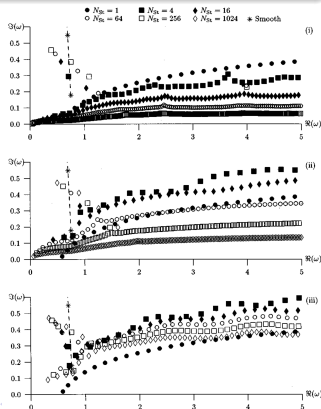
Test-bed study: Pöschl-Teller potential

From this we learn:

- Pseudospectrum contour lines seem to converge to QNM curves observed by Nollert [cf. Nollert 96].
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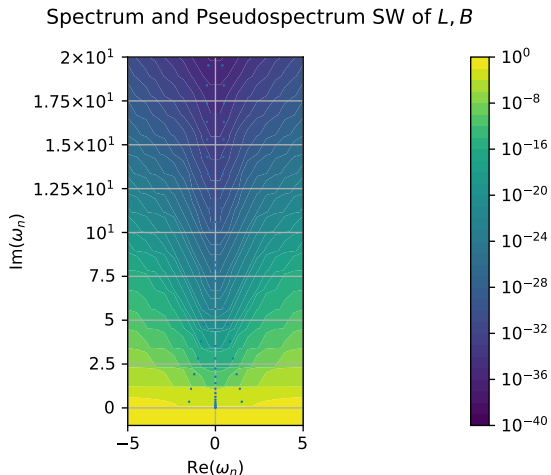


Curvature/w-modes in neutron stars [Kokkotas & Schmidt 99]



Schwarzschild QNMs

Schwarzschild Pseudospectrum: same qualitative behaviour (note: branch cut)



Highly damped QNMs unstable, slowest decaying QNMs stable.

QNM: (spherically symmetric) general case [JLJ, Macedo, Al Sheikh 21]

Starting point: (scalar) wave equation in “tortoise” coordinates

On a stationary spacetime (with timelike Killing ∂_t):

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V_\ell \right) \phi_{\ell m} = 0 ,$$

Dimensionless coordinates: $\bar{t} = t/\lambda$ and $\bar{x} = r_*/\lambda$ (and $\bar{V}_\ell = \lambda^2 V_\ell$),

Conformal hyperboloidal approach

$$\begin{cases} \bar{t} &= \tau - h(x) \\ \bar{x} &= f(x) \end{cases} .$$

- $h(x)$: implements the hyperboloidal slicing, i.e. $\tau = \text{const.}$ is a horizon-penetrating hyperboloidal slice Σ_τ intersecting future \mathcal{I}^+ .
- $f(x)$: spatial compactification between $\bar{x} \in [-\infty, \infty]$ to $[a, b]$.
- Timelike Killing: $\lambda \partial_t = \partial_{\bar{t}} = \partial_\tau$.

QNM: (spherically symmetric) general case [JLJ, Macedo, Al Sheikh 21]

First-order reduction: $\psi_{\ell m} = \partial_\tau \phi_{\ell m}$

$$\partial_\tau u_{\ell m} = iL u_{\ell m} \quad , \quad \text{with } u_{\ell m} = \begin{pmatrix} \phi_{\ell m} \\ \psi_{\ell m} \end{pmatrix}$$

where

$$L = \frac{1}{i} \left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array} \right)$$

$$L_1 = \frac{1}{w(x)} (\partial_x (p(x) \partial_x) - q(x)) \quad (\text{Sturm-Liouville operator})$$

$$L_2 = \frac{1}{w(x)} (2\gamma(x) \partial_x + \partial_x \gamma(x))$$

$$\text{with } w(x) = \frac{f'^2 - h'^2}{|f'|} > 0 \quad , \quad p(x) = \frac{1}{|f'|} \quad , \quad q(x) = |f'| V_\ell \quad , \quad \gamma(x) = \frac{h'}{|f'|}.$$

QNM: (spherically symmetric) general case [JLJ, Macedo, Al Sheikh 21]

Spectral problem

Taking Fourier transform, dropping (ℓ, m) (convention $u(\tau, x) \sim u(x)e^{i\omega\tau}$):

$$L u_n = \omega_n u_n .$$

where

$$L = \frac{1}{i} \left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array} \right)$$

$$L_1 = \frac{1}{w(x)} (\partial_x (p(x)\partial_x) - q(x)) \quad (\text{Sturm-Liouville operator})$$

$$L_2 = \frac{1}{w(x)} (2\gamma(x)\partial_x + \partial_x\gamma(x))$$

Conformal hyperboloidal approach: **No boundary conditions**

It holds $p(a) = p(b) = 0$, L_1 is “singular”: **BCs “in-built” in L .**

QNM: (spherically symmetric) general case [JLJ, Macedo, Al Sheikh 21]

Scalar product

Natural scalar product (where $\tilde{V}_\ell := q(x) > 0$):

$$\langle u_1, u_2 \rangle_E = \frac{1}{2} \int_a^b \left(w(x) \bar{\psi}_1 \psi_2 + p(x) \partial_x \bar{\phi}_1 \partial_x \phi_2 + \tilde{V}_\ell \bar{\phi}_1 \phi_2 \right) dx ,$$

associated with the “total energy” of ϕ on Σ_t , defining the “**energy norm**”

$$\|u\|_E^2 = \langle u, u \rangle_E = \int_{\Sigma_\tau} T_{ab}(\phi, \partial_\tau \phi) t^a n^b d\Sigma_\tau ,$$

Spectral problem of a non-selfadjoint operator

- Full operator L : not selfadjoint.
- L_2 : dissipative term encoding the energy leaking at \mathcal{I}^+ .
- L selfadjoint in the non-dissipative $L_2 = 0$ case.

Non-normal operators spectral tools: “energy norm” for Pseudospectrum.

QNM: (spherically symmetric) general case [JLJ, Macedo, Al Sheikh 21]

Key for later “missing degrees of freedom” discussion: Adjoint operator

$$L^\dagger = \frac{1}{i} \left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 + L_2^\partial \end{array} \right)$$

where

$$L_2^\partial = 2 \frac{\gamma}{w} \left(\delta(x-a) - \delta(x-b) \right)$$

Application to Pöschl-Teller

Conformal compactification

$$\begin{cases} \bar{t} = \tau - \frac{1}{2} \ln(1 - x^2) \\ \bar{x} = \operatorname{arctanh}(x) \end{cases} \Leftrightarrow \begin{cases} \tau = \bar{t} - \ln(\cosh \bar{x}) \\ x = \tanh \bar{x} \end{cases}$$

mapping $[-\infty, \infty]$ to $[a, b] = [-1, 1]$.

Spectral problem

Operators in L , with potential $V(x) = V_o \operatorname{sech}^2(\bar{x})$ (with $V_o = 1$):

$$L_1 = \partial_x ((1 - x^2) \partial_x) - 1$$

$$L_2 = -(2x \partial_x + 1) .$$

where:

$$w(x) = 1 \quad , \quad p(x) = (1 - x^2) \quad , \quad q(x) = \frac{V}{1 - x^2} =: \tilde{V}(x) \quad , \quad \gamma(x) = -x .$$

Application to Schwarzschild

Schwarzschild potential

Axial (Regge-Wheeler) case (also for polar (Zerilli) parity):

$$V_\ell^s = \left(1 - \frac{2M}{r}\right) \left(\frac{\ell(\ell+1)}{r^2} + (1-s^2)\frac{2M}{r^3}\right),$$

where

- $r_* = r + 2M \ln(r/2M - 1)$
- $s = 0, 1, 2$, respectively, to the scalar, electromagnetic and gravitational cases.

Numerical Chebyshev methods: analyticity of $V(x)$

Bizoń-Mach coordinates used in Pöschl-Teller not well adapted now: the potential is non-analytic in x , spoiling the accuracy of Chebyshev's methods.

Same problem for the polar (Zerilli) case.

Solution

We resort rather to the 'minimal gauge' hyperboloidal slicing [Ansorg, Macedo 16; Macedo 18] guaranteeing the analyticity of the Schwarzschild potential.

Application to Schwarzschild

Conformal compactification: “minimal gauge” [Ansorg, Macedo 16; Macedo 18]

$$\begin{cases} \bar{t} &= \tau - \frac{1}{2} (\ln \sigma + \ln(1 - \sigma) - \frac{1}{\sigma}) \\ \bar{x} &= \frac{1}{2} (\frac{1}{\sigma} + \ln(1 - \sigma) - \ln \sigma) \end{cases} ,$$

mapping $[-\infty, \infty]$ to $[a, b] = [0, 1]$ (we use σ , rather than x).

Spectral problem

Operators in L :

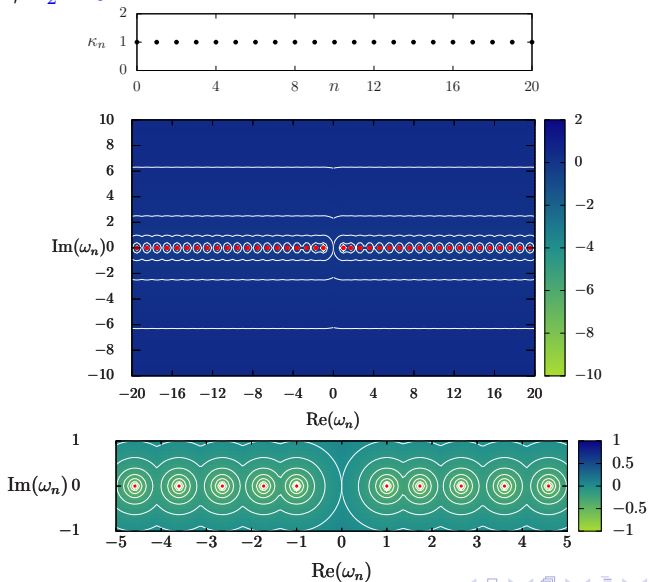
$$L_1 = \frac{1}{1 + \sigma} [\partial_\sigma (\sigma^2(1 - \sigma)\partial_\sigma) - (\ell(\ell + 1) + (1 - \sigma^2)\sigma)]$$

$$L_2 = \frac{1}{1 + \sigma} ((1 - 2\sigma^2)\partial_\sigma - 2\sigma) .$$

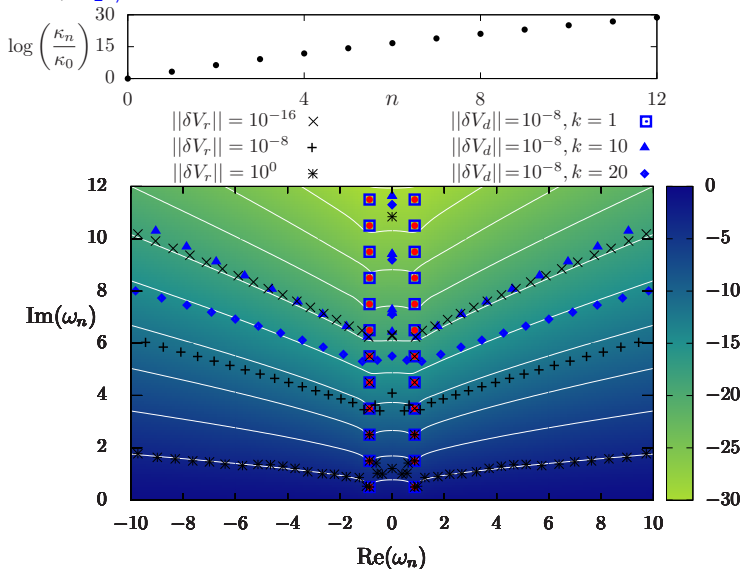
where:

$$w(\sigma) = 1 + \sigma , \quad p(\sigma) = \sigma^2(1 - \sigma) , \quad q(\sigma) = \frac{V}{\sigma^2(1 - \sigma)} =: \tilde{V}(\sigma) , \quad \gamma(\sigma) = 1 - 2\sigma^2 .$$

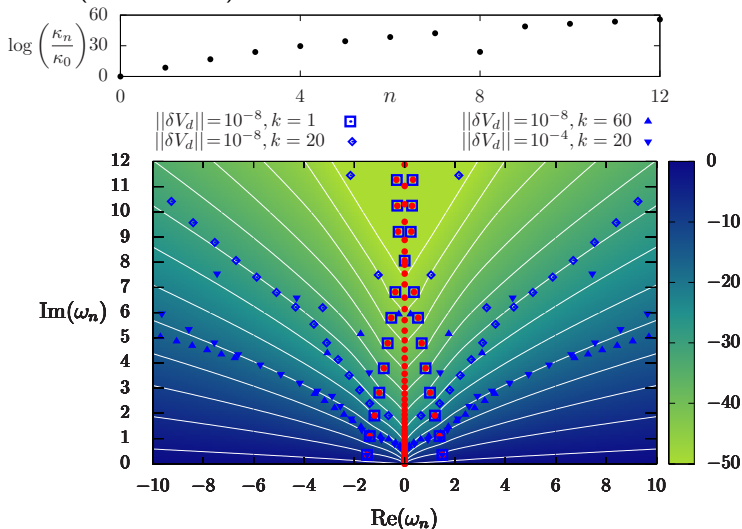
Trying to understand: the “current picture” ... in pictures

Pöschl-Teller, $L_2 = 0$:

Trying to understand: the “current picture” ... in pictures

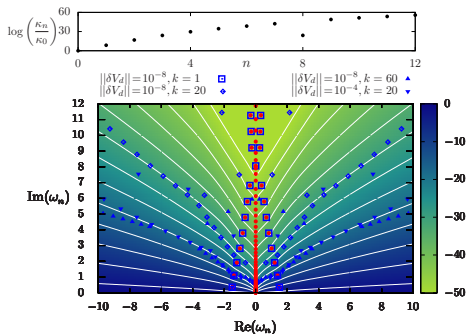
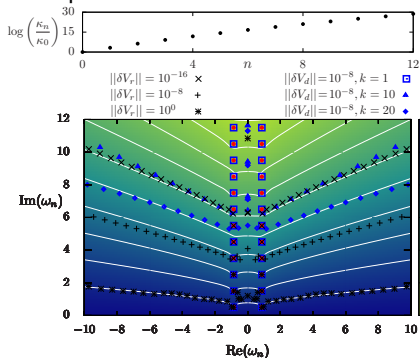
Pöschl-Teller, $L_2 \neq 0$:

Trying to understand: the “current picture” ... in pictures

Schwarzschild ($s = 2, \ell = 2$):

Trying to understand: the “current picture” ... in pictures

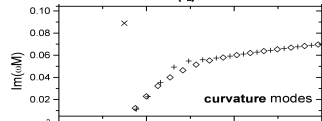
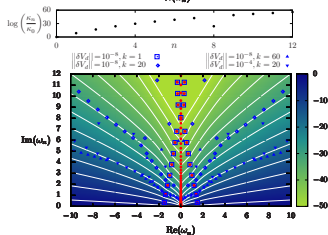
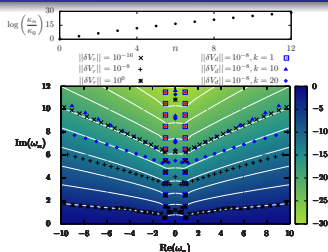
Comparison Pöschl-Teller versus Schwarzschild:



Remarks:

- High frequency perturbations: random δV_r , deterministic $\delta V_d \sim \cos(2\pi kx)$.
- **Fundamental QNM stable.**
- Perturbed QNMs “migrate” towards ϵ -contour lines of Pseudospectra.
- ‘Universality’ phenomenon?

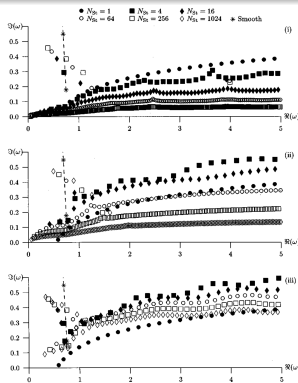
Trying to understand: the “current picture” ... in pictures



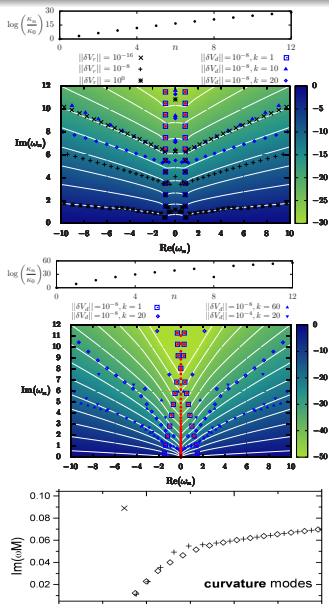
Black Hole and Neutron Star QNMs

Comparison with:

- Nollert's high-frequency Schwarzschild perturbations.
- Nollert's remark on Neutron Stars (w-modes) curvature QNMs.



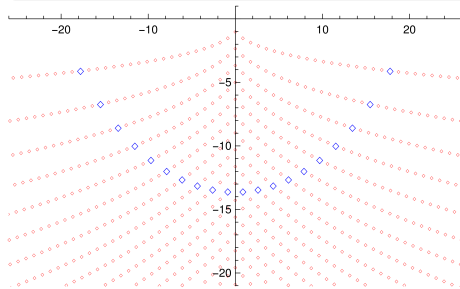
Trying to understand: the “current picture” ... in pictures



“Duality” QNMs (long-range potentials) and Regge poles (compact support potentials)?

QNM of a spherical obstacle [Stefanov 06]:

- Red-diamonds: fixed “ n ”, running angular ℓ .
- Blue-diamonds: fixed ℓ (here $\ell = 20$), running n .



Scheme

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- 5 Discussion, Conclusions and Perspectives: a low-regularity problem
 - Missing degrees of freedom and asymptotic symmetries

Connecting with Lax-Phillips: Keldysh expansion

Hilbert space: Keldysh's expansion of the resolvent [Keldysh 51, 71]

Given v_j and w_j (right- . left and right eigenvectors of L), with $\langle w_n, v_n \rangle = -1$, the resolvent of L can be expanded in a domain Ω around the poles as

$$(L - \lambda)^{-1} = \sum_{\lambda_j \in \Omega} \frac{|v_j\rangle\langle w_j|}{\lambda - \lambda_j} + H(\lambda) \quad , \quad \lambda \in \Omega \setminus \sigma(L)$$

QNM resonant expansions, “1st-order time reduction” [Al Sheikh, JLJ, Gasperin 21, 22]

The field u satisfying $\partial_\tau u = iLu$, with $u(\tau = 0, x) = u_0$, can be written in an “asymptotic expansion” as

$$u(\tau, x) = \sum_{j=1}^N e^{i\omega_j \tau} \kappa_j \langle \hat{w}_j | u_0 \rangle_E \hat{v}_j + E_N(\tau; S)$$

with $\|E_N(\tau; S)\|_E \leq C_N(a, L) e^{-a\tau} \|S\|_E$

Note the presence of the condition number κ_j .

Connecting with Lax-Phillips: Keldysh expansion

Hilbert space: Keldysh's expansion of the resolvent [Keldysh 51, 71]

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QNM resonant expansions, “scattered field” [Al Sheikh, JLJ, Gasperin 21, 22]

In terms of the scattered field satisfying the 2nd order equation

$$\phi(\tau, x) \sim \sum_n e^{i\omega_j \tau} a_n \hat{\phi}_j^R(x) \quad , \quad \phi(t, x) \sim \sum_n e^{i\omega_j t} a_n e^{i\omega_j h(x)} \hat{\phi}_j^R(x)$$

with $a_j = \frac{\kappa_j}{2} \left(\langle \hat{\phi}_j^L, \varphi_0 \rangle_{H^1_{(V,p)}} - i\omega_j \langle \hat{\phi}_j^L, \varphi_1 \rangle_{(2,w)} \right)$

that recovers the expression in terms of normal modes in the selfadjoint (normal) case: $\kappa_j = 1$, $\hat{\phi}_j^R(x) = \hat{\phi}_j^L(x)$.

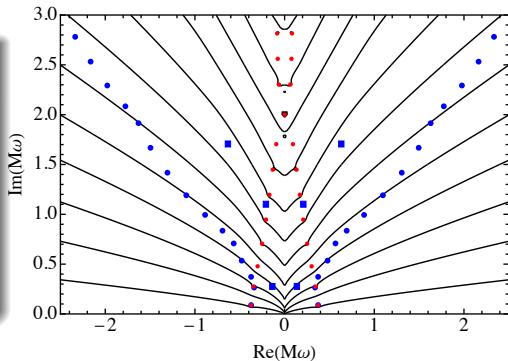
Are the perturbed QNMs in the ringdown signal? Yes

Extracting QNM: Mode Filtering

[JLJ, Macedo, Al Sheikh 21]

$$\Phi_{\text{spec}}^N(t) := \sum_{n=0}^N \mathcal{A}_n e^{i\omega_n t}$$

$$\mathcal{F}^N(t) = \Phi_{\text{evol}}(t) - \Phi_{\text{spec}}^N(t)$$



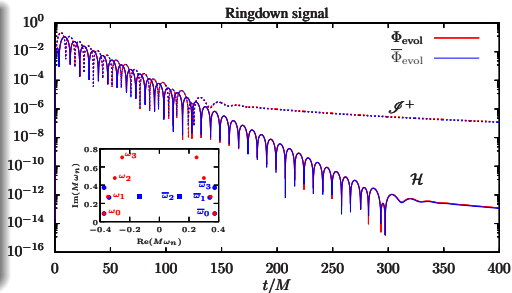
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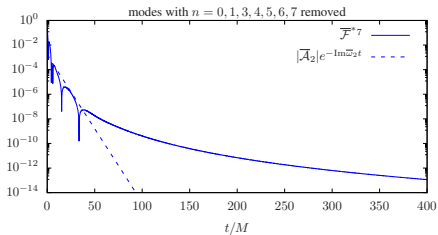
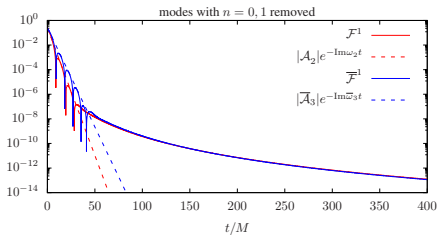
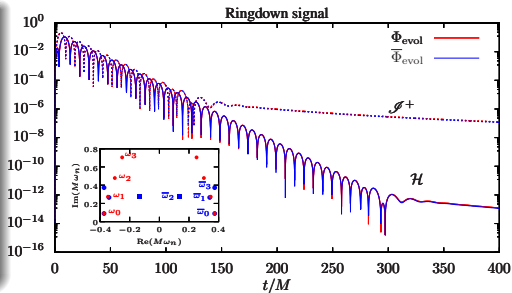
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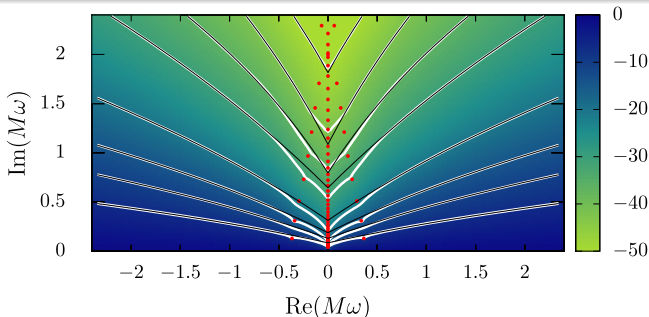
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Emerging picture: "Universality" of compact-object QNMs

Logarithm QNM-free regions: Pseudospectrum contour lines [JLJ, Macedo, Al Sheikh 21]

- QNM overtones: ultraviolet (high-frequency) unstable.
- Fundamental QNM: ultraviolet (high-frequency) stable.
(Warning: "Flea on the Elephant" effect for "trapping potentials" [Jona-Lasinio et al. 81, 84; Helffer & Sjöstrand 83, 85; Simon 85; Cheung et al. 21,...])
- Logarithmic $|\operatorname{Re}(\omega)| \gg 1$ asymptotics: $\operatorname{Im}(\omega) \sim C_1 + C_2 \ln(|\operatorname{Re}(\omega)| + C_3)$



Emerging picture: "Universality" of compact-object QNMs

 C^p regularity: Regge QNM branches, logarithmic asymptotics

- Compact support potential: **theorem** by Zworski (Weyl law) [Zworski 87]
(Note: this **fully** explains Nollert's results [Nollert 96])
- Non-compact support, $C^p < \infty$ potential: WKB Berry's treatment [Berry 82].
- Related results by Nollert-Price [Nollert & Price 99]
- Polytropic neutron stars: w -modes [Zhang, Wu & Leung 11]

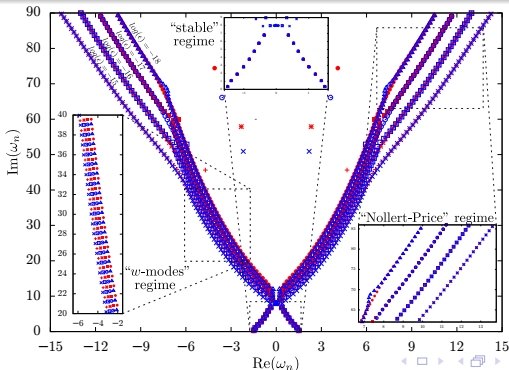
$$\begin{aligned} \operatorname{Re}(\omega_n) &\sim \pm \left(\frac{\pi}{L} n + \frac{\pi \gamma_p}{2L} \right) \\ \operatorname{Im}(\omega_n) &\sim \frac{1}{L} \left[\gamma \ln \left(\left(\frac{\pi}{L} n + \frac{\pi \gamma_p}{2L} \right) + \frac{\pi \gamma_\Delta}{2L} \right) - \ln S \right] \end{aligned}$$

- "Length scale" L , "Small structure" coefficient γ , "Strength coefficient" S .
- "Real shift" γ_Δ .
- "Parity term" γ_p .

Emerging picture: "Universality" of compact-object QNMs

Regular perturbations: "Nollert-Price-like" branches and "inner QNMs"

- Always above the pseudospectra log-lines: **BH QNM Isospectrality Loss.**
- "Nollert-Price-like": different branches, transitions at critical values.
- "Inner modes": appearing at the critical values. Weyl law.
- Nollert-Price branches towards logarithm pseudospectra as perturbation frequency increases: **Do they get to the logarithmic pseudospectra lines?**



From Burnett conjecture to a Regge QNM conjecture

Burnett's conjecture: high-frequency limit of spacetimes [Burnett 89; Huneau, Luk 18, 19]

The limit of high-frequency oscillations of a vacuum spacetime is effectively described by an effective matter spacetime described by massless Einstein-Vlasov.

Luk & Rodnianski: "null shells" as low-regularity problems in Einstein equations

- Spikes are more efficient in triggering instabilities [Gasperin & JLJ 21]:

$$|\delta g_{ab}| \|\delta L\|_E \sim \max_{\text{supp}(\delta g)} |\partial \delta g_{ab}|^2 \sim \max_{\text{supp}(\delta g)} \rho_E(\delta g; x)$$

- Luk & Rodnianski's treatment [Luk & Rodnianski 20]: Allowing for "concentrations", the infinite high-frequency limit is a **low regularity limit**

$$g_n \rightarrow g_\infty \text{ in } C^0 \cap H^1, \quad \partial g_n \rightarrow \partial g_\infty \text{ in } L^2$$

Regge QNM branches conjecture: a low-regularity problem [JLJ, Macedo, AL Sheikh 21, Gasperin & JLJ 21]

In the limit of infinite frequency, generic ultraviolet perturbations push BH QNMs in Nollert-Price branches to asymptotically logarithmic branches along the QNM-free region boundary, exactly following the Regge QNM asymptotic pattern.

QNMs and norms: “Definition” versus “Stability” problems

Definition of QNMs: we can (need to) choose the norm to control QNMs

- Ansorg & Macedo [Ansorg & Macedo 16]: if C^∞ -regularity, then every point in the upper complex plane is an eigenvalue. Need of “more control”.
- Warnick & Gajic [Warnick & Gajic 19]: Fundamental contribution by identifying in **Gevrey-2** regularity classes in (**asymptotically flat**) hyperboloidal framework (complemented in [Galkowski & Zworski 20] in a complex scaling setting).
Hilbert spaces of $(\sigma, 2)$ -Gevrey functions on $[0, \rho_0]$: $f, g \in C^\infty((0, \rho_0); \mathbb{C})$

$$\langle f, g \rangle_{G_{\sigma, 1, \rho_0}^2} = \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{n!^2 (n+1)!^2} \int_0^{\rho_0} \partial_\rho^n f \partial_\rho^n \bar{g} d\rho$$

Stability of QNMs: we cannot choose the norm, the “physics” chooses for us

The system chooses what is a “big and small” perturbation: **here, the energy.**

$$\|(\phi, \psi)\|_E = E = \int_{\Sigma} T_{ab} t^a n^b d\Sigma \sim \|\phi\|_{H_V^1} + \|\psi\|_{L^2}$$

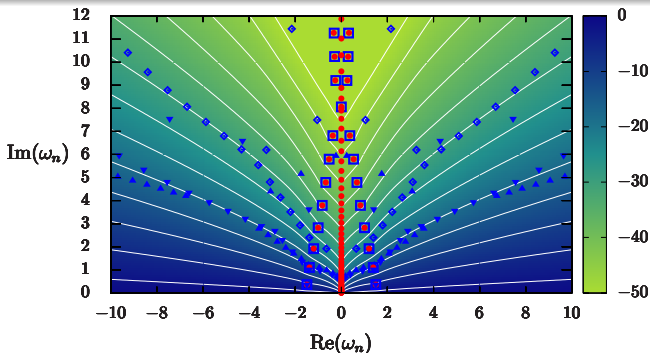
“Energy-Pseudospectrum = Chart of the Gevrey Ocean”

QNMs and norms: “Definition” versus “Stability” problems

Towards a reconciliation: the Gevrey Ocean

A small high-frequency perturbation in the energy, is a huge one in Gevrey terms.

- For a given V , QNM eigenfunctions are defined to be in class Gevrey-2.
- Small (in the energy norm) perturbations δV of V lead to very different QNM ω_n 's, respective eigenfunctions being indeed Gevrey-2.



“Energy-Pseudospectrum = Chart of the Gevrey Ocean”

Plan

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- 2 Non-normal operators: spectral instability and Pseudospectrum
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The instability “is not” (only) in the potentials...

QNM instability and Pseudospectrum codes

- Pseudospectrum non-trivial for $V = 0$.
- Pseudospectrum trivial for $V = 0$ and $L_2 = 0$.

Then:

- Potential instability is encoded in the hyperbolic structure, namely in null asymptotic boundaries.
- In stationary BHs and in Minkowski: frozen degrees of freedom (but phase space is large and is there...)

Towards a geometric description of QNMs

Compactified hyperboloidal slicing: geometric data

- i) Choose a slicing $\{\mathcal{S}_\tau^\infty\}$ of \mathcal{I}^+ . Given the generator k^a along \mathcal{I}^+ consider, at each slice \mathcal{S}_τ^∞ , the only null normal ℓ^a to \mathcal{S}_τ^∞ satisfying $k^a \ell_a = -1$. Repeat on the BH horizon.

- ii) Choose a function γ on null infinity sections \mathcal{S}_τ^∞ (respectively, also a function γ on $\mathcal{S}_\tau^{\mathcal{H}}$ in BH spacetimes) and consider the null vector $\gamma^a = \gamma \ell^a$, outgoing from \mathcal{M} . Repeat on the BH horizon.

- iii) Extend the vector γ^a arbitrarily to the bulk (under appropriate conditions).

- iv) Slices Σ_τ in $\tilde{\mathcal{M}}$ are compact manifolds with boundary, with boundaries given by spheres.

- v) Resulting data:

$$(\{\Sigma_\tau\}, \gamma_{ab}, K_{ab}, w; \{\mathcal{S}_\tau^\infty\}, \{\mathcal{S}_\tau^{\mathcal{H}}\}, \gamma)$$

Evolution equations

$$\partial_\tau u = iLu \quad , \quad u = \begin{pmatrix} \phi \\ \psi = \partial_\tau \phi \end{pmatrix} \quad , \quad L = \frac{1}{i} \left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array} \right)$$

$$L_1 = \frac{1}{w} (\tilde{\Delta} - \tilde{V}(x)), \quad L_2 = \frac{1}{w} (2\gamma \cdot \tilde{D} + \tilde{D} \cdot \gamma)$$

Adjoint infinitesimal evolution operator

$$L^\dagger = L + L^\partial \quad , \quad L^\partial = \frac{1}{i} \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & L_2^\partial \end{array} \right)$$

$$L_2^\partial = -2 \frac{\gamma^a k_a}{w} (\delta_{S\mathcal{H}} - \delta_{S\infty}) = -2 \frac{(\gamma k^a) \ell_a}{w} (\delta_{S\mathcal{H}} - \delta_{S\infty})$$

Missing degrees of freedom and BMS symmetry

Non-selfadjointness and missing degrees of freedom

- **Non-selfadjointness in dynamical problems generically reflects that some degrees of freedom are missing or are being lost:** the system is not isolated or is part of a larger system, leading to non-conservative dynamics.
- To cast such systems in terms of a selfadjoint problem, one must “complete” the system by adding an appropriate set of degrees of freedom.

how could we render our BH perturbation problem selfadjoint?

- In our scattering case, degrees of freedom flow to infinity and through the **BH** horizon.
- Consider adding formal degrees of freedom at future null infinity \mathcal{I}^+ and at the BH horizon \mathcal{H} , in such a way that they account for the degrees of freedom and energy leaving the system.
- Placing “*Geiger* counters” at the infinite and horizon boundaries, such that the total number of degrees of freedom is conserved.

Coupling boundary and bulk degrees of freedom

Being anchored to the boundary, such boundary degrees of freedom would be non-propagating ones (in the bulk)

- Flux at boundaries:

$$F(\tau) = \int_{S^{\mathcal{H}}} \gamma |\partial_\tau \phi|^2 dS + \int_{S^\infty} \gamma |\partial_\tau \phi|^2 dS$$

Bondi-Sachs flux of energy-momentum at null infinity if: i) γ is built from the $\ell = 0, 1$ spherical harmonics and ii) we identify $\partial_\tau \phi$ as a news function \mathcal{N} .

- Here, γ arbitrary function on S_τ^∞ : supertranslation in the asymptotic BMS group.

$$\xi^a = \gamma k^a, \quad \mathcal{L}_k \gamma = 0$$

corresponds to a supertranslation in the asymptotic BMS group.

- Flux term (arbitrary $\gamma(x)$) ["Coupling" action: BCs as action boundary terms (Roberto!!!)]:

$$H_\xi = \int_{\mathcal{I}^+} (\gamma \partial_\tau \phi \partial_\tau \phi + \gamma \partial_\tau \phi + \partial_\tau \phi \Delta_{S^\infty} \gamma) d\tau dS$$

corresponding to the **BMS supertranslation ξ^a and generating symplectomorphisms in the (appropriate) phase space of degrees of freedom that "live on the boundary"**.

Missing degrees of freedom and BMS symmetries

A “heuristic” proposal

Missing degrees of freedom can be encoded in BMS supertranslation degrees of freedom:

- i) **BMS supertranslations ξ^a would stand as a dynamical symmetry**, acting on (and generating) the **phase space of degrees of freedom on the boundary** (the “clicks” in the Geiger counter analogy), with γ a **degree of freedom living on the null boundaries**.
- ii) The whole geometric construction starts from the **choice of slicing of null boundaries**. This is **arbitrary**, but all choices are **related by an appropriate BMS supertranslation**.
- iii) **The set of degrees of freedom (ϕ, γ) would be complete**. By this we mean that, as ϕ flows away, degrees of freedom γ are “activated” through a coupling controlled by the Hamiltonian (1), so **a total energy of the type $E_o = E(\phi, \psi) + H_\xi$ would be conserved**.

Missing degrees of freedom and BMS symmetries

“Hopes” :

- From the perspective of the extended space of degrees of freedom (ϕ, γ) , one could gain **insight into the observed instability of BH QNMs**.

Paraphrasing this in terms of inverse scattering theory: **additional BMS degrees of freedom could provide necessary additional data at the boundaries** —complementary to (transmission and) reflection coefficients and (possible) bound energy levels— needed to determine the scattering potential.

- Conservative formulation of the dynamics can be constructed in the extended space of degrees of freedom (ϕ, γ) , then a notion of **global spacetime normal modes**, as eigenfunctions of the time evolution generator, could be envisaged. This could be of **interest both from a phenomenological and from a fundamental (quantization) perspective**.

Conclusions and Perspectives

Conclusions

- Numerical evidence of **instability of QNM overtones** under high-frequency perturbations in the effective potential: **Nollert-Price BH QNM branches**.
Fundamental QNM ultraviolet stable.
- Independent support from **Pseudospectrum** of non-perturbed operator.
- Hints towards **Universality of compact-object QNM branches** in the infinite high-frequency limit: a low regularity problem.

Others:

- **ϵ -dual QNM expansions**: from “energy-stability” of time-domain signal.
 - i) Keldysh expansion: on non-perturbed Kerr QNMs.
 - ii) Keldysh expansion: on perturbed BH QNMs.
- **BH QNM isospectrality loss**: patterns.
- ...

Conclusions and Perspectives

Perspectives

- **Can we measure the regularity of spacetime?**

$$\operatorname{Re}(\omega_n) \sim \pm \left(\frac{\pi}{L} n + \frac{\pi \gamma_P}{2L} \right)$$

$$\operatorname{Im}(\omega_n) \sim \frac{1}{L} \left[\gamma \ln \left(\left(\frac{\pi}{L} n + \frac{\pi \gamma_P}{2L} \right) + \frac{\pi \gamma_\Delta}{2L} \right) - \ln S \right]$$

**A (universal) ultraviolet-instability BH QNM conjecture:
Regge QNM branches in the infinite frequency limit.**

- **ϵ -dual QNM expansions and BH spectroscopy:**
 - i) On non-perturbed Kerr QNMs: “average” BH background properties.
 - ii) On perturbed BH QNMs: properties of perturbations (astrophysical perturbations, **small-scale fundamental gravity physics...**)
- **Proofs:** from heuristic and numerical evidences to **Theorems**.
- Towards a **fully geometric picture** for BH QNM instability: **BMS charges**.

Conclusions and Perspectives

Vishveshwara, C.V., "On the black hole trail...: a personal journey"

18th Conference of the Indian Association for General Relativity and Gravitation, pp. 11-22 (1996)

wave during the coalescence of binary black holes[18]. Recently Aguirregabiria and I have studied the sensitivity of the quasinormal modes to the scattering potential[19]. The motivation is to understand how any perturbing influence, such as another gravitating source, that might alter the effective potential would thereby affect the quasinormal modes. Interestingly, we find that the fundamental mode is, in general, insensitive to small changes in the potential, whereas the higher modes could alter drastically. The fundamental mode would therefore carry the imprint of the black hole, while higher modes might indicate the nature of the perturbing source.

Quasinormal modes are perhaps the rebuttal to the criticism of my thesis examiner regarding the nonobservability of black holes.

Conclusions and Perspectives

The “Gevrey Ocean” and the Dijon Legacy...



Conclusions and Perspectives

“Truth suffers from much Analysis”

(Ancient Fremen saying) Dune Messiah, Frank Herbert



A 'hint of Geometry' perhaps...?