Extreme horizons and peeling

Meudon, 12-14 December 2022

- This is a joint project with Jack Borthwick (Besançon) and Eric Gourgoulhon (Paris observatory)
- Origin of the project: observation that some fields have, near an extreme horizon, a behaviour analogous to near an asymptotically flat infinity, provided one uses an adequate set of coordinates.

More details: Jack during his thesis under my supervision studied by spectral methods the scattering of Dirac fields by an extreme Kerr-de Sitter black hole. The scattering was described by so-called wave operators that compare the full evolution with simplified dynamics in asymptotic regions. Near the extreme horizon, using a tortoise coordinate, the simplified dynamics was generated by the hamiltonian (neglecting short and long-range perturbations)

$$i\gamma^0\gamma^1\frac{\partial}{\partial r_*} + i\frac{c}{r_*}\gamma^0\gamma^2$$

which is the radial part of the Dirac equation on flat space-time (using r_* instead of r as the radial variable and after separation of variables).

Also the crossing sphere, that exists for single horizons, is pushed away at infinite spacelike distance from the exterior and is replaced by an "internal infinity".

Hence the question : can we study peeling near an extreme horizon and are the initial data giving rise to the peeling behaviour at the horizon in a similar class to what we get for Minkowski or Schwarzschild?

Upon investigating the question, we came across something called the Couch - Torrence inversion, which makes this analogy precise in the case of the extreme Reissner - Nordström spacetime.

This is work in progress and some of it is tentative.

Let me start by saying a few words about the peeling

1. Peeling for the wave equation on Schwarzschild's spacetime

It is an asymptotic behaviour of massless fields along outgoing null geodesics, first described by Sachs in 1961 in terms of expansion of the field in powers of $\frac{1}{r}$ and alignement of the principal null directions of the terms of the expansion along the null geodesic. In 1965, Roger Penrose proved that it is equivalent to a much simpler property : the continuity of the rescaled field at $\mathscr{I}$.

Lionel Mason and I studied the peeling on Schwarzschild in 2007 using a characterisation of the regularity at $\mathscr{I}$ in terms of Sobolev spaces. Using energy estimates, we determined the classes of initial data for the wave equation that would give rise to a rescaled field regular at order k at $\mathscr{I}^+$. We showed that the classes of data for Minkowski and Schwarzschild are the same at all orders.

Two main features of our approach:

→ partial compactification $\Omega = \frac{1}{r}$
  i^0 remains at infinity

→ choice of observer: based on the requirements
  - strong enough energy at $\mathscr{I}^+$ (transverse to $\mathscr{I}^+$)
  - good behaviour of the Killing form for $\Omega^2 g$ at $\mathscr{I}^+$ and i^0.

$\Longrightarrow$ Morawetz vector field.

How would one address the question of the peeling at an extreme horizon?

1. No compactification required
2. The essential question is to find an adequate vector field

- A few details

Schwarzschild metric in coordinates (u, R, ω) rescaled by R^2

$$\hat{g} = R^2 g = R^2(1-2MR)\,du^2 - 2\,du\,dR - d\omega^2$$

$\mathscr{I}^+$ described as $\{R=0\} \times \mathbb{R}_u \times S^2_\omega$

$$\mathrm{Scal}_{\hat{g}} = 12MR$$

On $\hat{g}$, we study the conformal wave equation

$$\boxed{\square_{\hat{g}} \phi + \frac{1}{6} \mathrm{Scal}_{\hat{g}} \phi = 0}$$

- Morawetz vector field

On Minkowski spacetime: conformal Killing vector field

$$T = u^2 \partial_u + v^2 \partial_v \qquad u = t - r, \; v = t + r$$

$$= u^2 \partial_u - 2(1 + uR) \partial_R \qquad u = t - r, \; R = \frac{1}{r}$$

η Minkowski metric, $\hat{\eta} = R^2 \eta \qquad \mathcal{L}_T \hat{\eta} = 0$

We keep the second form and use it in our case

$$\boxed{T = u^2 \partial_u - 2(1 + uR) \partial_R}$$

$$\boxed{\mathcal{L}_T \hat{g} = 4MR^2 (3 + uR) \, du^2}$$

We work on

$$\Omega_{u_0} = \{ t \geq 0 \} \cap \{ u \leq u_0 \} \qquad \text{for } u_0 << -1 \text{ given}$$

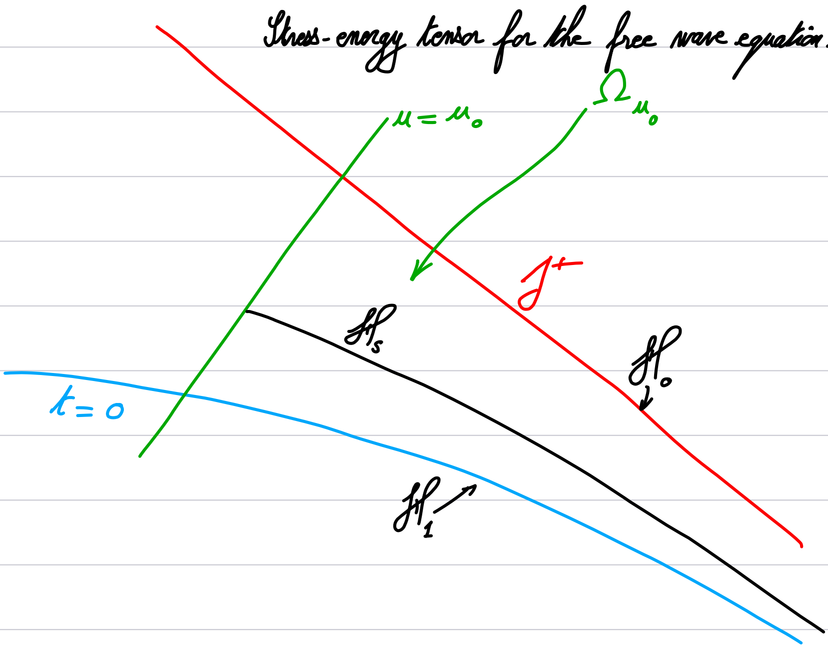
Foliation

$$\mathcal{H}_s = \{ u = -s r_* \} \qquad 0 < s \leq 1$$

$s = 1 \qquad \mathcal{H}_1 = \{ t = 0 \} \cap \{ u \leq u_0 \}$

$s \to 0 \qquad \mathcal{H}_0$ understood as $\mathcal{J}^+ \cap \{ u \leq u_0 \}$

Stress-energy tensor for the free wave equation.



We define function spaces on $\mathcal{H}_s$ using the energy associated with T (T timelike and future oriented in Ω_{u_0}). Increase regularity by commuting partial derivatives $\partial_u, \partial_R, \nabla_{S^2}$

$\Rightarrow$ get estimates both ways using simply Gronwall estimates

$\Rightarrow$ This gives us the exact classes of data that produce a field that is regular to order k at $\mathcal{J}^+$.

When comparing these classes to the ones obtained on Minkowski, we find that they are the same

2. The Couch-Torrence inversion

- This is described in the literature as a conformal isometry of the exterior of an extreme Reissner-Nordström black hole, that exchanges the horizon and infinity.

- A remarkable property is that there is a choice of conformal scale for which the Couch-Torrence inversion is an actual isometry of the spacetime (not present in the literature to our knowledge). In fact many choices but one is particularly useful.

- Usual presentation: Couch-Torrence GRG 1984
Bizon-Friedrich CQG 2013
Aretakis Springer book 2018

$$\boxed{g = \left(\frac{r-M}{r}\right)^2 dt^2 - \left(\frac{r}{r-M}\right)^2 dr^2 - r^2 d\omega^2}$$

Put $\boxed{z = \dfrac{Mr}{r-M}}$

$r \to M^+ \iff z \to +\infty$
$z \to M^+ \iff r \to +\infty$

$$\Rightarrow \quad g = \left(\frac{M}{z-M}\right)^2 \left(\left(\frac{z-M}{z}\right)^2 dt^2 - \left(\frac{z}{z-M}\right)^2 dz^2 - z^2 d\omega^2 \right)$$

Hence $\Phi(t,z,\omega) = (t,r,\omega)$ is a conformal isometry of g and

$$\Phi^* g = \left(\frac{z-M}{M}\right)^2 g = \left(\frac{M}{r-M}\right)^2 g$$

Note: $z = \dfrac{Mr}{r-M} \iff r = \dfrac{Mz}{z-M}$
so the transformation is an involution.

- Also introduce r_* such that $\dfrac{dr_*}{dr} = \left(\dfrac{r}{r-M}\right)^2$
and r_* vanishes at the photon sphere $r = 2M$

$$r_* = r - M + 2M \log\left(\frac{r-M}{M}\right) - \frac{M^2}{r-M}$$

Then $r = \dfrac{Mz}{z-M}$ gives $r - M = \dfrac{M^2}{z-M}$

$$r_* = r - M + 2M\log\left(\frac{r-M}{M}\right) - \frac{M^2}{r-M}$$
$$= -\left(z - M + 2M\log\left(\frac{z-M}{M}\right) - \frac{M^2}{z-M}\right)$$
$$= -z_*$$

where $\dfrac{dz_*}{dz} = \left(\dfrac{z}{z-M}\right)^2$ and $z_* = 0$ for $z = 2M$.

$\boxed{\text{So the Couch-Torrence inversion simply means changing } r_* \text{ into } -r_* .}$

However, if instead of g we work with

$$\hat{g} = R^2 g = \frac{1}{r^2} g$$
$$= R^2(1-mR)^2(dt^2 - dr_*^2) - d\omega^2$$

Applying the Couch-Torrence inversion

$$r_* = -z_*$$

$$R = \frac{1}{r} = \left(\frac{Mz}{z-M}\right)^{-1} = \frac{z-M}{Mz} = \frac{1}{M}(1 - MZ)$$

$$Z = \frac{1}{z}$$

$$1 - MR = MZ$$

Hence

$$\hat{g} = R^2(1-MR)^2(dt^2 - dr_*^2) - d\omega^2$$
$$= Z^2(1-MZ)^2(dt^2 - dz_*^2) - d\omega^2$$

We see that $\boxed{\Phi^* \hat{g} = \hat{g}}$ (1)

2. Peeling at the extreme Reissner-Nordström horizon

(1) entails that we can characterise completely the peeling at the horizon at any given order in terms of classes of initial data at $t=0$. These classes are obtained from those at infinity through the Couch-Torrence inversion. As this inversion is merely $r_* \longrightarrow -r_*$, the classes at the horizon and at infinity are the same in terms of regularity and fall-off when expressed using r_*.

$\Longrightarrow$ All we need is to reproduce our work in Schwarzschild in the extreme Reissner-Nordström case. This works.

$$T = u^2 \partial_u - 2(1+uR)\partial_R$$

$$\boxed{\mathcal{L}_T \hat{g} = \left(-8R^3M^2 + 12R^2M - 4(R^4M^2 - R^3M)u\right) du^2}$$

3. Extension to more general situations (Couch-Torrence inversion with a loose end)

A fairly natural idea is to apply the Couch-Torrence inversion to a general "extreme horizon". We do not expect it to be a global isometry of the space-time, rather we consider it as a diffeomorphism between two different spacetimes, one that is a neighbourhood of an extreme horizon and the other a neighbourhood of an infinity of some description.

The general form of a metric near an extreme horizon is:

$$g = (r-M)^2 F(r,x)\, dv^2 - 2\, dv\, dr - 2(r-M)\, \ell_\alpha(r,x)\, dv\, dx^\alpha - \gamma_{\alpha\beta}(r,x)\, dx^\alpha dx^\beta$$

in so-called gaussian null coordinates adapted to the horizon described here as $r = M$. The function F is positive and does not vanish at $r = M$. We consider the following spherically symmetric situation:

$$\boxed{g = (r-M)^2 F(r)\, dv^2 - 2\, dv\, dr - r^2 d\omega^2} \quad (2)$$

where F is analytic and positive on $[M, r_1[$.

Note that here M is just the value of a coordinate at the horizon and does not have any particular physical meaning.

We perform the Couch-Torrence inversion on g

$$r = \frac{Mz}{z-M}$$

$$\Rightarrow g = \left(\frac{M}{z-M}\right)^2 \left(M^2 F(r)\, dt^2 - \frac{1}{M^2 F(r)} dz^2 - z^2 d\omega^2\right) \quad (3)$$

defined in the neighbourhood of $z = +\infty$.

- What is the asymptotic geometry of g as $z \to +\infty$?

Put $\alpha = M\sqrt{F(M)}$

$$g = \left(\frac{M}{\alpha(z-M)}\right)^2 \left(\alpha^2 M^2 F(r)\, dt^2 - \frac{\alpha^2}{M^2 F(r)} dz^2 - \alpha^2 z^2 d\omega^2\right)$$

Change t into $\tilde{t} = \alpha^2 t$ and put $\Psi\left(\frac{1}{z}\right) = \frac{MF(r)}{\alpha^2}$

$$g = \left(\frac{M}{\alpha(z-M)}\right)^2 \left(\Psi\left(\frac{1}{z}\right) dt^2 - \frac{1}{\Psi\left(\frac{1}{z}\right)} dz^2 - \alpha^2 z^2 d\omega^2\right)$$

Ψ is analytic on an interval of the form $[0, a]$, positive and $\Psi(0) = 1$.

$$\Rightarrow \tilde{g} = \left(\frac{\alpha(z-M)}{M}\right)^2 g$$

has an asymptotic structure of the form

$$\tilde{g}_\infty = dt^2 - dr^2 - \alpha^2 r^2 d\omega^2$$

In the extreme Reissner-Nordström case we have $\alpha = 1$, $\tilde{g}$ is the Minkowski metric and $\tilde{g}$ is asymptotically flat.
$\mathcal{I}_\infty$

In general we cannot assume that $\alpha = 1$ and $\tilde{g}$ is asymptotically "conical". The question is whether one can establish peeling results at infinity for g in general and whether the classes of data giving rise to a peeling at a given order will be comparable to those on Minkowski spacetime.

4. Peeling at a general spherically symmetric degenerate horizon.

- First we compactify the metric at $z = \infty$

To do this, we start with a rescaled metric

$$\hat{g} = R^2 g \quad , \quad R = \frac{1}{r} = \frac{z - M}{Mz}$$

$$\begin{aligned}\hat{g} &= R^2 g \\ &= \frac{(z-M)^2}{M^2 z^2} \frac{M^2}{(z-M)^2} \left(M^2 F(r) dt^2 - \frac{1}{M^2 F(r)} dz^2 - z^2 d\omega^2 \right) \\ &= \frac{1}{z^2} \left(M^2 F(r) dt^2 - \frac{1}{M^2 F(r)} dz^2 - z^2 d\omega^2 \right)\end{aligned}$$

So putting $Z = \frac{1}{z}$, $h(z) = M^2 F(r)$,

$$\hat{g} = Z^2\left(h(z)\,dt^2 - \frac{1}{h(z)}dz^2\right) - d\omega^2$$

Define z_* and $u(t,z)$ by

$$h(z)\,dz_* = dz \quad , \quad u = t - z_*$$

and put

$$f\left(\frac{1}{z}\right) = h(z) \qquad f \text{ analytic and } > 0 \text{ on } [0,a]$$

$$\hat{g} = Z^2 f(Z)\,du^2 - 2\,du\,dZ - d\omega^2$$

- Now forget about where this comes from and about the unusual variables and study the peeling for the metric

$$\boxed{\hat{g} = R^2 f(R)\,du^2 - 2\,du\,dR - d\omega^2}$$

$\mathscr{I}^+$ described as $\{R=0\} \times \mathbb{R}_u \times S^2_\omega$

$$\mathrm{Scal}_{\hat{g}} = -R^2 f''(R) - 4Rf'(R) + 2(1 - f(R))$$

it does not vanish at $\mathscr{I}^+$ unless $f(0) = 1$.

On $\hat{g}$, we study the conformal wave equation

$$\boxed{\Box_{\hat{g}}\,\phi + \frac{1}{6}\,\mathrm{Scal}_{\hat{g}}\,\phi = 0}$$

- Morawetz vector field

$$\boxed{T := u^2 \partial_u + 2(1+uR)\partial_R}$$

$$\boxed{\mathcal{L}_T \hat{g} = \left(4R(1-f(R)) - 2R^2 f'(R) - 2R^2 u f'(R)\right) du^2}$$

In the Schwarzschild case we had

$$\mathcal{L}_T \hat{g} = 4MR^2(3+uR)\,du^2$$

We have a weaker decay here due to $f(0) \neq 1$ and $f'(0) \neq 0$ but this is analogous to what we had in the extreme Reißner-Nordström case. We have enough decay to establish the peeling using exactly the same method.

$\Longrightarrow$ The weaker asymptotic flatness does not modify the conditions for peeling

In fact, we have a peeling for massive fields as well!

5_ Projects

- Extend the study to completely general degenerate horizons.

The peeling for the Kerr metric works essentially as in the Schwarzschild case (JPN, Pham Truong Xuan, 2017?) so one would expect a similar behaviour.

Gaussian null coordinates

$\mathcal{N}$ null hypersurface generated by l null (integral curves of l are null geodesics)

Let n null defined on $\mathcal{N}$ such that $g(l,n) = 1$.

Let H be a spacelike hypersurface of $\mathcal{N}$ such that

cross-section

$$\mathcal{N} \simeq \mathbb{R} \times H \quad , \quad H = \mathcal{N} / \text{integral lines of } l$$

Choose coordinates x^α on H. Let u be the parameter along the integral curves of l set to 0 at H.
Let r be the affine parameter on the null geodesics with initial direction n (or $-n$, we choose the vector that points inside the spacetime) set to H at $\mathcal{N}$.

$\Rightarrow$ this defines coordinates in a tubular neighbourhood of $\mathcal{N}$

$$(u, r, x^\alpha)$$