Cartan Geometries modeled on the lightlike cone in Minkowski space-time

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Some personal remarks

- My original motivation was the intrinsic geometry of lightlike hypersurfaces in Lorentzian manifolds.
- On the contrary to semi-Riemannian geometry, there is no linear connection, there is no notion of curvature ... and no differential operators...
- I bought the Sharpe book here in Paris several years ago. Sharpe constructed Riemannian, conformal and projective geometries à la Cartan. That is, starting from a homogeneous model and a Cartan connection.
- The concept of a generalized space was introduced by Élie Cartan in order to build a bridge between geometry in the sense of Felix's Klein Erlangen program and Differential Geometry[1] (Riemann).
- Is it possible to apply these ideas to lightlike manifolds?

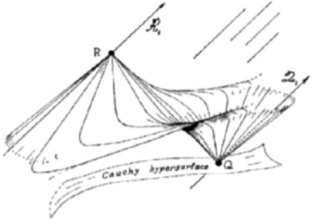
  My approach contains open questions and, it is still far from being final.

[1] A. Čap and J. Slovák, Parabolic Geometries I. Background and General Theory, Mathematical Surveys and Monographs **154**, AMS 2009.

OUTLINE

1. Definitions, examples and more ...

1. Lightlike manifolds
2. Carrollian geometries
3. Examples



Lightlike manifolds (intrinsic definition)

A lightlike manifold is a triple (N^{m+1}, h, Z) where

1. h is a symmetric $(0,2)$-tensor on the manifold N^{m+1} which is positive but non-definite and whose radical

$$\mathrm{Rad}\, h = \{v \in TN : h(v, -) = 0\}$$

defines a 1-dimensional distribution on N^{m+1}.

2. $Z \in \Gamma(\mathrm{Rad}\, h) \subset \mathfrak{X}(N)$ spans the radical distribution at every point.

$\boxed{\mathrm{Aut}(N^{m+1}, h, Z)}$ may be infinite dimensional.

Examples of lightlike manifolds

- Lightlike hypersurfaces of time-oriented Lorentzian manifolds.
- In General Relativity a space-time $(\widetilde{M}, \widetilde{g})$ is called asymtotically simple, if there is another Lorentzian manifold (M, g) such that
  1. $\widetilde{M}$ is an open subset of M with smooth boundary $\partial\widetilde{M} = \mathcal{I}$.
  2. There is $\Omega \in C^{\infty}(M, \mathbb{R})$ such that $g = \Omega^2\widetilde{g}$ on $\widetilde{M}$ and $\Omega \mid_{\mathcal{I}} = 0$ but $d\Omega \neq 0$ on $\mathcal{I}$.
  3. Every inextendible future (past) lightlike geodesic of $(\widetilde{M}, \widetilde{g})$ has future (past) endpoint on $\mathcal{I}$.

  Then,

$$\left(\mathcal{I}, g\mid_{\mathcal{I}}, Z := (d\Omega)^{\sharp}\right) \text{ is a lightlike manifold.}$$

Carrollian geometries (Levy-Leblond, 1965)

A lightlike manifold (N^{m+1}, h, Z) is a Carrollian geometry when the space of lines N^{m+1}/Z is a manifold Σ^m and the projection

$$\pi\colon N^{m+1} \to \Sigma^m$$

is a fiber bundle. We write $(\pi\colon N^{m+1} \to \Sigma^m, h, Z)$.

The mathematician Charles Lutwidge Dodgson, under the pseudonym Lewis Carroll, wrote Alice's aventures in Wonderland in 1865.

Example of Carrollian geometry

The bundle of scales of a conformal Riemannian manifold (M, c)

1. $$\pi \colon \mathcal{L} = \{g_x : g \in c, x \in M\} \to M$$

   is a principal fiber bundle with structure group $\mathbb{R}_+$ ($g_x \bullet t := t^2 g_x$).
2. The tautological metric

   $$h(\xi, \eta) = g_x(T_{g_x}\pi \cdot \xi, T_{g_x}\pi \cdot \eta), \quad \xi, \eta \in T_{g_x}\mathcal{L},$$
3. $$Z_{g_x} = \frac{d}{dt}\,|_{t=0}\,(e^{2t} g_x).$$

$(\pi \colon \mathcal{L} \to M, h, Z)$ is a Carrollian geometry.

Every section $s \in \Gamma(\pi)$ gives by pull-back of h a Riemannian metric in the conformal class c and conversely.

Lightlike manifolds typically provide Carrollian geometries

- Let (N^{m+1}, h, Z) be a lightlike manifold.
- The space of orbits N^{m+1}/Z is typically a manifold Σ^m (the absolute space)

$$\pi\colon N^{m+1} \to \Sigma^m := N^{m+1}/Z.$$

- The vector field Z can be assumed to be complete and then we have the action (suppose that integral curves of Z are lines)

$$N^{m+1} \times \mathbb{R}_+ \to N^{m+1}, \quad (y, t) \mapsto y \cdot t = \mathrm{Fl}^Z_{\log t}(y)$$

and $\pi\colon N^{m+1} \to \Sigma^m$ becomes an $\mathbb{R}_+$-principal fiber bundle.
- $(\pi\colon N^{m+1} \to \Sigma^m, h, Z)$ is a Carrollian geometry.

Carroll geometries as generalized bundles of scales

Assume $\pi \colon N^{m+1} \to \Sigma^m$.

$$\Big(\underbrace{T_y N/\mathrm{Rad}(h_y)}_{\mathcal{E}_y}, h_y\Big) \xrightarrow{T_y\pi} \Big(T_{\pi(y)}\Sigma, c(y)\Big) \quad \Longrightarrow \quad c(y) \in \mathrm{Sym}^+(T_{\pi(y)}\Sigma)$$

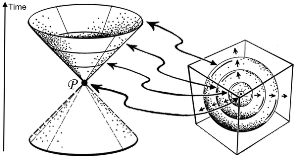
$$\begin{array}{ccc} N & \xrightarrow{c} & \mathrm{Sym}^+(T\Sigma) \\ \pi \searrow & & \swarrow \\ & \Sigma & \end{array}$$

Every section s of π gives the Riemannian metric $s^* h = c \circ h$ on Σ.

- $\mathcal{L}_Z h = 0 \quad \Longrightarrow$ h induces a Riemaniann metric on Σ.
- $\mathcal{L}_Z h = 2\rho h \, (\rho \neq 0) \Longrightarrow$ h induces a Riemaniann conformal structure on Σ.

2. The lightlike cone in Minkowski spacetime: the model

1. The lightlike cone as homogeneous space
2. The lightlike cone as Carrollian geometry
3. The Lie algebras level description
4. Why the lightlike cone?



R. Penrose

Our model space ...

Let $\mathbb{L}^{m+2}$ be the Minkowski space-time with basis $(\ell, e_1, \cdots, e_m, \eta)$ such that the Lorentzian metric $\langle\ ,\ \rangle$ corresponds to the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathrm{I}_m & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The lightlike cone is the hypersurface

$$C^{m+1} := \left\{ v \in \mathbb{L}^{m+2} : \langle v, v \rangle = 0, \quad v \neq 0 \right\}$$

- C^{m+1} is a lightlike manifold with $h = \langle\ ,\ \rangle$ and Z the restriction of the position vector field.
- C^{m+1} is also a Carrollian geometry

$$\pi \colon C^{m+1} \to C^{m+1}/Z \simeq \mathbb{S}^m$$

... a refinement of the model space.

- The Lorentz group $O(m+1,1)$ acts transitively on C^{m+1} and preserves h.
- $O(m+1,1)$ also acts on $\mathbb{S}^m = C^{m+1}/Z$

$$O(m+1,1)\times\mathbb{S}^m\to\mathbb{S}^m,\quad (g,[x])\mapsto[g\cdot x]$$

  preserving the conformal class of the round metric. This action is not effective.
- In order to get an effective action, we replace the Lorentz group with the Möbius group :

$$G = PO(m+1,1) := O(m+1,1)/\{\pm\mathrm{Id}\}$$

  and C^{m+1} with $\boxed{\mathcal{N}^{m+1} := C^{m+1}/\mathbb{Z}_2}$.
- We identify antipodal points in C^{m+1} and then, h and Z induce a lightlike metric and a vector field which spans its radical.

$\mathcal{N}^{m+1}$ as homogeneous space of Möbius group

The action of the Lorentz group descends to an action of the Möbius group

$$G \times \mathcal{N}^{m+1} \to \mathcal{N}^{m+1}, \quad [\sigma] \cdot [v] := [\sigma \cdot v]$$

which is also transitive and preserves h and Z.

Therefore

$$\mathcal{N}^{m+1} = G/H,$$

where $H \subset G$ is the isotropy group of $[\ell] = \{\pm \ell\} \in \mathcal{N}^{m+1}$.

- Our *model* for lightlike manifolds

$$\boxed{\left(\mathcal{N}^{m+1} = G/H,\ h,\ Z\right)}$$

The Carrollian geometry of $\mathcal{N}^{m+1}$. The absolute space for $\mathcal{N}^{m+1}$

- $\pi\colon \mathcal{N}^{m+1} \to \mathcal{N}^{m+1}/Z \simeq \mathbb{S}^m$, the space of lightlike lines in $\mathcal{N}^{m+1}$.
- $\mathbb{S}^m = G/P$ is the model for conformal geometry, $\boxed{\mathrm{Conf}(\mathbb{S}^m) = G.}$
  $P \subset G$ is the isotropy group of $\pi[\ell] \in \mathbb{S}^m$ (Poincaré conformal group)

$$P = \left\{ \begin{bmatrix} \lambda & -\lambda w^t g & -\frac{\lambda}{2}\langle w, w\rangle \\ 0 & g & w \\ 0 & 0 & \lambda^{-1} \end{bmatrix} : \lambda \in \mathbb{R} \setminus \{0\}, w \in \mathbb{R}^m, g \in O(m) \right\}$$

- The isotropy group of $[\ell] \in \mathcal{N}^{m+1}$ is

$$H = \{\sigma \in P : \lambda = \pm 1\} \cong \mathbb{R}^m \rtimes O(m) = \mathrm{Euc}(\mathbb{R}^m)$$

$$\boxed{G/H = \mathcal{N}^{m+1} \xrightarrow{\pi} \mathbb{S}^m = G/P, \quad gH \mapsto gP.}$$

$\mathcal{N}^{m+1} = G/H$ at Lie algebras level

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & Z & 0 \\ X & A & -Z^t \\ 0 & -X^t & -a \end{pmatrix} : a \in \mathbb{R}, X \in \mathbb{R}^m, Z \in (\mathbb{R}^m)^*, A \in \mathfrak{so}(m) \right\}$$

$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $\mathbb{Z}$-grading, that is, $\boxed{[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}}$

$\boxed{\mathfrak{g} = \underbrace{\mathfrak{g}_{-1} \oplus \mathfrak{z}(\mathfrak{g}_0)}_{\mathfrak{m}} \oplus \underbrace{[\mathfrak{g}_0, \mathfrak{g}_0] \oplus \mathfrak{g}_1}_{\mathfrak{h}}}$ and $\boxed{\mathfrak{h} = \underbrace{[\mathfrak{g}_0, \mathfrak{g}_0]}_{\mathfrak{so}(m)} \oplus \mathfrak{g}_1}$ lightlike model

$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ is not a reductive decomposition!!

Even more, the Lie algebra $\mathfrak{h}$ does not admit any reductive complement in $\mathfrak{g}$.

$\boxed{\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1}$ and $\boxed{\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1}$ conformal model

Why the lightlike cone?

1. The relationship between lightlike manifolds and conformal geometry
$$G/H = \mathcal{N}^{m+1} \xrightarrow{\pi} \mathbb{S}^m = G/P.$$
$\mathcal{N}^{m+1}$ is the bundle of scales of $(\mathbb{S}^m, c)$.
2. $\boxed{\mathrm{Aut}(N^{m+1}, h, Z)}$ may be infinite dimensional but ...

Theorem[a]

- For $m \geq 2$, $\boxed{\mathrm{Aut}(\mathcal{N}^{m+1}, h, Z) = G := PO(m+1, 1) = \mathrm{Con}(\mathbb{S}^m).}$
- For $m \geq 3$, every automophism between two connected open subsets of $\mathcal{N}^{m+1}$ is the restriction of the left translation by an element of G. (Liouville type Theorem).
- If $m = 2$, the (global) automophisms group of $\mathcal{N}^3$ is also isomorphic to $G = \mathrm{Conf}(\mathbb{S}^2)$ but the group of local isometries of $\mathcal{N}^3$ is the group of local conformal transformations of $\mathbb{S}^2$.

[a] E. Bekkara, C. Frances and A. Zeghib, Actions of semisimple Lie groups preserving a degenerate Riemannian metric, *Trans. Amer. Math. Soc.* **362** (5), (2010), 2415-2434.

3. Cartan geometries

$$\begin{array}{ccc} \textit{Riemannian Geometry} & \xrightarrow{\textit{generalization}} & \textit{Cartan Geometries} \\ \textit{generalization}\uparrow & & \uparrow\textit{generalization} \\ \textit{Euclidean Geometry} & \xrightarrow[\textit{generalization}]{} & \textit{Klein Geometries (Homogeneous spaces)} \end{array}$$

This diagram is taken from Sharpe's book.

Let $p : \mathcal{P} \to M$ be an H-principal fiber bundle.

- For every $X \in \mathfrak{h}$, the fundamental vector field $\zeta_X \in \mathfrak{X}(\mathcal{P})$ is $\zeta_X(u) := \frac{d}{dt}\,|_{t=0}\,(u \cdot \exp(tX))$.

Cartan geometry of type (G, H) on M (Charles Ehresmann, 1950)

- An H-principal fiber bundle $p : \mathcal{P} \to M$.
- A one-form $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$, called the Cartan connection such that
  1. $\omega(u)(\zeta_X(u)) = X$ for each $X \in \mathfrak{h}$.
  2. $(r^h)^*\omega = \mathrm{Ad}(h^{-1}) \circ \omega$ for all $h \in H$.
  3. $\omega(u) : T_u\mathcal{P} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{P}$.

$$\boxed{\dim(\mathcal{P}) = \dim(G), \quad \dim(M) = \dim(G/H)}$$

The curvature form

- $K \in \Omega^2(\mathcal{P}, \mathfrak{g}), \quad K = d\omega + \frac{1}{2}[\omega, \omega].$

- $(G \to G/H, \omega_G)$ is called the homogeneous model for Cartan geometries of type (G, H) where ω_G is the Maurer-Cartan form of G given by

$$\omega_G(g) : T_g G \to \mathfrak{g}, \quad L_X(g) \mapsto X, \quad X \in \mathfrak{g}.$$

The Maurer-Cartan equation

$$d\omega_G + \frac{1}{2}[\omega_G, \omega_G] = 0$$

implies the homogeneous model has zero curvature.

- The converse is locally true. *The curvature measures how far is our Cartan geometry from being the homogeneous model.*

The group of automophisms of $(p : \mathcal{P} \to M, \omega)$

$$\mathrm{Aut}(\mathcal{P},\omega) = \Big\{(F, f) : \text{automorphism of } p : \mathcal{P} \to M \text{ with } F^{*}\omega = \omega\Big\}$$

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{F} & \mathcal{P} \\ p\downarrow & & \downarrow p \\ M & \xrightarrow[f]{} & M \end{array} \qquad \begin{cases} F(u \cdot h) = F(u) \cdot h, \ \ u \in \mathcal{P}, h \in H. \\ F \text{ is a diffeomorphism} \\ F^{*}\omega = \omega. \end{cases}$$

When M is a connected manifold,

$$\boxed{\mathrm{Aut}(\mathcal{P},\omega) \text{ is a Lie group and } \dim \mathrm{Aut}(\mathcal{P},\omega) \leq \dim G.}$$

The tangent bundle of a Cartan geometry of type (G, H)

For each $u \in \mathcal{P}$ with $p(u) = x \in M$, there is a canonical linear isomorphism ϕ_u such that the following diagram conmutes

$$\begin{array}{ccc} T_u\mathcal{P} & \xrightarrow{\omega(u)} & \mathfrak{g} \\ {\scriptstyle T_u p}\downarrow & & \downarrow{\scriptstyle \rho} \\ T_x M & \xrightarrow[\phi_u \cong]{} & \mathfrak{g}/\mathfrak{h} \\ & & \Updownarrow \\ & & T_o(G/H) \end{array} \quad \text{with } \phi_{uh} = \underline{\mathrm{Ad}}(h^{-1})\phi_u \text{ for all } h \in H.$$

where $\underline{\mathrm{Ad}}$ is the representation of H given by

$$\underline{\mathrm{Ad}} : H \to \mathrm{Gl}(\mathfrak{g}/\mathfrak{h}), \quad h \mapsto \underline{\mathrm{Ad}}(h)(X + \mathfrak{h}) = \mathrm{Ad}(h)(X) + \mathfrak{h}.$$

Every *geometric structure* on $\mathfrak{g}/\mathfrak{h}$ invariant under $\underline{\mathrm{Ad}}$ is carried to M.

Correspondence spaces

- Let G be a Lie group and let $H \subset P \subset G$ be closed subgroups.

For a Cartan geometry $(p : \mathcal{P} \to M, \omega)$ with model (G, P), we define the correspondence space $\mathcal{C}(M)$ of M for the closed subgroup $H \subset P$ to be the quotient space $\mathcal{P}/H$:

$$\begin{array}{ccc} G & & \\ \downarrow & \searrow & \\ G/H & \longrightarrow & G/P \end{array} \quad \Rightarrow \quad \begin{array}{ccc} & \mathcal{P} & \\ & \downarrow & \searrow \\ \mathcal{P}/H = & \mathcal{C}(M) & \longrightarrow \; M = \mathcal{P}/P \end{array}$$

The projection $\pi : \mathcal{C}(M) \to M$ is a fiber bundle with fiber P/H and

$(\pi : \mathcal{P} \to \mathcal{C}(M), \omega)$ is a Cartan geometry of type (G, H).

4. Cartan geometries with model the lightlike cone

Theorem I (P*, 21)

Every Cartan geometry $(p : \mathcal{G} \to N^{m+1}, \omega)$ of type $\mathcal{N}^{m+1} = G/H$ determines a lightlike geometry $(N^{m+1}, h^{\omega}, Z^{\omega})$ and

$$\mathcal{G} \simeq \Big\{ (Z^{\omega}_y, e_1, \cdots, e_m) \in \mathcal{P}^1 N^{m+1} : h^{\omega}(e_i, e_j) = \delta_{ij} \Big\}.$$

($\mathcal{G}$ gives a G-structure on N^{m+1} with structure group H.)

$$\begin{array}{ccc} T_u\mathcal{G} & \xrightarrow{\omega(u)} & \mathfrak{g} \\ {\scriptstyle T_u p}\downarrow & & \downarrow{\scriptstyle \rho} \\ T_y N^{m+1} & \xrightarrow[\phi_u \cong]{} & \mathfrak{g}/\mathfrak{h} \end{array}$$

$$\mathbb{R} \oplus \mathbb{R}^m \cong T_{[\ell]}\mathcal{N}^{m+1} \cong \mathfrak{g}/\mathfrak{h}, \quad q\left(\begin{pmatrix} a \\ x \end{pmatrix}, \begin{pmatrix} b \\ y \end{pmatrix} \right) = \langle x, y \rangle, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathfrak{X}(\mathfrak{g}/\mathfrak{h})$$

$$\boxed{\underline{\mathrm{Ad}}(h) \begin{pmatrix} a \\ x \end{pmatrix} = \begin{pmatrix} a - \langle x, gw \rangle \\ gx \end{pmatrix}, \quad h \leftrightarrow (w, g) \in \mathbb{R}^m \rtimes O(m) = H.}$$

Let $(p : \mathcal{G} \to N^{m+1}, \omega)$ be a Cartan geometry of type $\mathcal{N}^{m+1} = G/H$.

$$\mathcal{H} := \omega^{-1}(\underbrace{\mathfrak{g}_{-1} \oplus \mathfrak{z}(\mathfrak{g}_0)}_{\mathfrak{m}}) \subset T\mathcal{G}$$

defines general connection (horizontal distribution) on $p\colon \mathcal{G} \to N^{m+1}$.

Theorem II (P*, 21)

$$\mathrm{Aut}(\mathcal{G}, \omega) = \Big\{ f \in \mathrm{Aut}(N^{m+1}, h^\omega, Z^\omega) : F = Tf \text{ preserves the distribution } \mathcal{H} \Big\}.$$

That is, $T_u F \cdot \mathcal{H}(u) = \mathcal{H}(F(u))$.

$$F\colon \mathcal{G} \to \mathcal{G}, \quad u = (Z^\omega_y, e_1, \cdots, e_m) \mapsto F(u) = (Z^\omega_{f(y)}, T_y f \cdot e_1, \cdots, T_y f \cdot e_m)$$

The bundle of scales $(\pi\colon \mathcal{L} \to M^m, h, Z)$ of a conformal Riemannian structure (M^m, c)

- For $m \geq 3$, there is an equivalence between conformal Riemannian structures on M and normal Cartan geometries $(p\colon \mathcal{P} \to M^m, \omega)$ of type (G, P).

  The correspondence space $\mathcal{C}(M)$ for $H \subset P$ is $\mathcal{C}(M) \simeq \mathcal{L}$ and there is a Cartan geometry

  $$(\mathcal{P} \to \mathcal{L}, \omega)$$

  of type $\mathcal{N}^{m+1} = G/H$ such that $h^\omega = h$ (the tautological metric) and $Z^\omega = Z$.

- For $m = 2$, there is an equivalence between Möbius structures $(M^2, c = [g], p)$

  $$p\colon c \to \mathcal{T}_{(0,2)}(M^2) \Leftrightarrow \begin{cases} p(g) \text{ is symmetric .} \\ \mathrm{trace}_g\, p(g) = K^g. \\ p(e^{2\sigma} g) = P^g - \frac{1}{2}\|\nabla^g \sigma\|_g^2 g - \mathrm{Hess}^g(\sigma) + d\sigma \otimes d\sigma. \end{cases}$$

  and Cartan geometries $(p\colon \mathcal{P} \to M^2, \omega_p)$ of type (G, P).

  The Möbius structure plays the role of the Schouten tensor for $m = 2$.

  For every Möbius structures on M^2 there is a Cartan geometry

  $$(\mathcal{P} \to \mathcal{L}, \omega)$$

  of type $\mathcal{N}^3 = G/H$ such that $h^\omega = h$ and $Z^\omega = Z$.

Let

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \mathfrak{g}$$

be the grading element of $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$

Proposition

Let $(p : \mathcal{G} \to N^{m+1}, \omega)$ be a Cartan geometry of type $\mathcal{N}^{m+1} = G/H$. Then $(N^{m+1}, h^\omega, Z^\omega)$ is locally isomorphic to the bundle of scales of a conformal Riemannian structure (M, c) if and only if the curvature of ω satisfies

$$K(\omega^{-1}(E), *) = 0.$$

Schwarzschild exterior and interior and Reissner-Nordström

Warped product space-times with Lorentzian two dimensional base

(B, g_B) a Lorentz surface, (F^m, g_F) a Riemann manifold and $0 < \lambda \in C^\infty(B, \mathbb{R})$.

$$(M^{m+2}, g) = (B \times_\lambda F, g := g_B + \lambda^2 g_F)$$

Assume $\xi \in \mathfrak{X}(B)$ is lightlike, then the distribution $\xi^\perp \subset TM^{m+2}$ is integrable

- M^{m+2} is foliated by lightlike manifolds.
- Let $\alpha \colon I \to B$ be an integral curve of ξ. Then

$$N^{m+1} = \Big\{(\alpha(s), x) : s \in I, x \in F\Big\} \subset M^{m+2}$$

is one of these lightlike manifolds.

$$(\pi_F\colon N^{m+1}\to F, h=g\,|_{N^{m+1}}, Z) \text{ is a Carrollian geometry.}$$

- For every $y=(\alpha(s),x)\in N^{m+1}$,

$$\Big(T_yN^{m+1}/\mathrm{Rad}(h_y), h_y\Big) \xrightarrow{T_y\pi_F} \Big(T_xF^m, c(y)\Big) \implies \quad c(y)=\lambda(\alpha(s))^2 g_F$$

$$\implies \qquad \begin{array}{ccc} N^{m+1} & \xrightarrow{c} & \mathcal{L}\subset \mathrm{Sym}^+(TF) \\ \pi_F \searrow & & \swarrow \\ & F^m & \end{array}$$

The pull-back by c of the tautological metric on $\mathcal{L}$ is $h=g\,|_{N^{m+1}}$.
Assume λ is injective along the curve α.

There is a Cartan geometry[2] on N^{m+1} of type $\mathcal{N}^{m+1}=G/H$ such that $h^\omega=h$.

[2] for $m=2$, we need a Möbius structure on F^2.

4-dimensional exterior Schwarzschild space-time

1.
$$g_B = -h(r)dt^2 + \frac{1}{h(r)}dr^2, \quad h(r) = \frac{1}{1 - \frac{2\mathbf{M}}{r}}$$
on the open subset $B = \{(t, r) \in \mathbb{R}^2 : r > 2\mathbf{M}\}$.

2. $(F^m, g_F) = (\mathbb{S}^2, g_{\mathbb{S}^2})$ and $\lambda(r) = r$.

3. $\xi = \frac{1}{h}\partial_t + \partial_r \in \mathfrak{X}(B)$ is a lightlike vector field.
$$\alpha(s) = \left(t_0 + s + 2\mathbf{M}\log\left(1 + \frac{s}{r_0 - 2\mathbf{M}}\right), r_0 + s\right), \quad s > 2\mathbf{M} - r_0.$$

4.
$$N^3 = \left\{(\alpha(s), x) : s > 2\mathbf{M} - r_0, x \in \mathbb{S}^2\right\} \subset M^4 \xrightarrow{c} \mathcal{N}^3$$

5.
$$\begin{cases} N^3 \simeq \{t^2 g_{\mathbb{S}^2}(x) : x \in \mathbb{S}^2, t > 2\mathbf{M}\} \subset \mathcal{N}^3 \subset \mathbb{L}^4. \\ p(g_{\mathbb{S}^2}) = \frac{1}{2} g_{\mathbb{S}^2}. \end{cases}$$

Remaining questions

1. Study other examples in physically realistic space-times.
2. Describe $\mathrm{Aut}(\mathcal{G}, \omega)$ in terms of the base manifold N^{m+1}.

$$\mathrm{Aut}(\mathcal{G}, \omega) = \mathrm{Aut}(N^{m+1}, h^{\omega}, Z^{\omega}, \underbrace{\cdots\cdots ???}_{\text{additional tensors...}})$$

3. Is there an equivalence one-to one between a family of lightlike geometries and certain class of Cartan geometries with this model?
4. Develop the general theory of Cartan connections with model the lightlike cone $\mathcal{N}^{m+1} = G/H$.
   - Tractor bundles ...
   - Tractor connections ...
   - ...

Thank you very much for your attention!!

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