SHEAVES, STACKS, AND SHTUKAS

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In this course, we first discuss various types of sheaves on adic spaces. One might to hope to have a theory of quasicoherent sheaves corresponding to arbitrary topological modules over Huber rings, but this is too optimistic; we will limit our ambitions to considering sheaves associated to finitely generated modules. We first review the properties of coherent sheaves in rigid analytic geometry, using a reduction procedure introduced by Tate to limit the need for local calculations to a particularly simple setting; in the process, we obtain Huber's generalization to strongly noetherian adic spaces. Unfortunately, without noetherian hypotheses this argument breaks down, but we will still obtain a well-defined theory of locally free coherent sheaves (a/k/a vector bundles) associated to a large class of spaces, including perfectoid spaces. (One can extend this construction from finite projective modules to *pseudocoherent* modules, which admit unbounded projective resolutions by finite projective modules.)

We next consider a particular class of adic spaces associated to perfectoid rings; in the case of perfectoid fields, these are the (adic) *Fargues-Fontaine (FF) curves*, which turn out to be strongly noetherian and hence admit a sensible theory of coherent sheaves. This theory has strong echoes with the theory of vector bundles on algebraic curves; for example, the semistable vector bundles are related to p-adic Galois representations in much the same way that semistable vector bundles on Riemann surfaces are related to unitary representations (Narasimhan-Seshadri theorem). This theory also gives rise to an analogue of the theory of *shtukas* used in the study of the Langlands correspondence over function fields; roughly speaking, a shtuka is a vector bundle with some additional marked local structure.

We finally introduce some ideas from the theory of algebraic stacks, with an eye towards the construction of moduli spaces of vector bundles and shtukas. The key point is to represent spaces using functors defined only on perfectoid spaces of characteristic p (these functors have come to be called *diamonds*); for example, the FF curves appear naturally as "absolute products" in the category of functors. While rigid analytic spaces give rise to such functors, this loses some information in a manner we will quantify. We will stop short of giving any formal discussion about precise "stacky" definitions or constructions of moduli spaces, we will articulate some statements about families of FF curves, and sheaves thereon, that translate into basic geometric prooerties of moduli spaces.

The projects will focus on vector bundles on FF curves, and combinatorial properties of their slope filtrations and slope polygons.

- Determine the precise relationship between the slope polygon of a filtered bundle and that of its associated graded bundle.
- Produce examples of families exhibiting specific degenerations of slope polygons consistent with semicontinuity.

- Depending on participant background, extend some of these results to "vector bundles with G-structure" where G is a reductive algebraic group over $\mathbb{Q}_p$ (the case $G = \mathrm{GL}_n(\mathbb{Q}_p)$ corresponding to vector bundles of rank n with no additional structure). In this case the slope polygon is naturally replaced by an invariant defined by Kottwitz carrying a finite amount of additional data.

References

These references represent primary sources for the material in the lectures. It is not a list of assumed prerequisites! More targeted suggestions for background reading will be included in the detailed lecture notes.

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[3] L. Fargues and J.-M. Fontaine, Courbes et fibrés vectoriels en théorie de Hodge p-adique, in preparation; draft (September 2015) available at `http://webusers.imj-prg.fr/~laurent.fargues/`.

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[7] R. Kottwitz, Isocrystals with additional structure, *Compos. Math.* **56** (1985), 201–220.

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