Math 308L - Autumn 2017
Final Exam
December 14, 2017

KEY

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
6	12	
7	12	
Total:	84	

- There are 7 problems on this exam. Be sure you have all 7 problems on your exam.
- The final answer must be left in exact form. Box your final answer.
- You are allowed the TI-30XIIS calculator. It is possible to complete the exam without a calculator.
- You are allowed a single sheet of 2-sided self-written notes.
- You must show your work to receive full credit. A correct answer with no supporting work will receive a zero.
- Use the backsides if you need extra space. Make a note of this if you do.
- Do not cheat. This exam should represent your own work. If you are caught cheating, I will report you to the Community Standards and Student Conduct office.

Conventions:

- I will often denote the zero vector by 0.
- When I define a variable, it is defined for that whole question. The A defined in Question 2 is the same for each part.
- I often use x to denote the vector $(x_1, x_2, \ldots, x_n)$. It should be clear from context.
- Sometimes I write vectors as a row and sometimes as a column. The following are the same to me.

$$(1,2,3) \qquad \begin{bmatrix}1\\2\\3\end{bmatrix}.$$

- I write the evaluation of linear transformations in a few ways. The following are the same to me.

$$T(1,2,3) \quad T((1,2,3)) \quad T\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right)$$

1. Give an example of each of the following. If it is not possible, write "NOT POSSIBLE".

(a) (2 points) Give an example of a 2×3 matrix A and a vector $b \in \mathbb{R}^2$ such that $Ax = b$ has no solutions but $Ax = 0$ has infinitely many solutions.

Solution: Let
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$
and $b = (0, 1)$.

(b) (2 points) Give an example of a linear system in 3 variables whose solution space is the intersection of the $x + y + z = 0$ plane and the xy-plane.

Solution: The linear system given by
$$\begin{aligned} x + y + z &= 0 \\ z &= 0 \end{aligned}$$

(c) (2 points) Give an example of a 2×2 matrix A such that $A^4 = I_2$ but $A^2 \neq I_2$. If possible, give the matrix A explicitly.

Solution: Let A be the rotation by $\pi/2$ matrix. This is given by
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

(d) (2 points) Give an example of 2 linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\text{range}(T) = \ker(S)$.

Solution: Let $T(x, y) = (x, y)$ and $S(x, y) = (0, 0)$.

(e) (2 points) Give an example of an orthogonal matrix that is not invertible.

Solution: NOT POSSIBLE. The inverse of an orthogonal matrix is its transpose.

(f) (2 points) Give an example of an diagonalizable matrix that is not orthogonally diagonalizable.

Solution:
$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

2. Let A be defined by

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) (4 points) Find a basis for the solution space $Ax = 0$.

Solution: $\{(2, -1, 0)\}$

(b) (4 points) What is the general solution to $Ax = \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}$?

Solution: $(6, 0, -3) + s_1(2, -1, 0)$.

(c) (4 points) Is there a vector $y \in \mathbb{R}^3$ such that $Ax = y$ has no solutions? If so, give an example. If not, why not?

Solution: Yes. Many possibilities.

3. Let A and B be equivalent matrices defined by

$$A = \begin{bmatrix} -3 & 3 & -1 & -9 & 3 \\ 2 & -2 & 1 & 7 & -1 \\ 4 & -4 & 5 & 23 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 2 & -2 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

(a) (4 points) Find a basis for the solution space of $Ax = 0$.

Solution: $\{(1, 1, 0, 0, 0), (-2, 0, -3, 1, 0), (2, 0, -3, 0, 1)\}$

(b) (4 points) Let a_1, a_2, a_3, a_4, a_5 be the columns of A. Define $C = [a_1\ a_2\ a_3\ a_4]$. What is a particular solution to $Cx = a_5$?

Solution: $(-2, 0, 3, 0)$.

(c) (4 points) Using the same variables as (b), what is the general solution to $Cx = 3a_4 - a_5$?

Solution: $(8, 0, 6, 0) + s_1(1, 1, 0, 0) + s_2(-2, 0, -3, 1)$.

4. Let S be a subspace of $\mathbb{R}^4$ defined by

$$S = \operatorname{span}\left\{ \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix} \right\}.$$

(a) (3 points) What is a basis for $(S^{\perp})^{\perp}$?

Solution: $\{(1,1,0,0),(0,0,1,1)\}$.

(b) (3 points) What is a basis for $S^{\perp}$?

Solution: $\{(1,-1,0,0),(0,0,1,-1)\}$.

(c) (3 points) Does there exist a rank 2 matrix A such that $\operatorname{null}(A) = S$? If so, give an example. If not, why not?

Solution: If $\operatorname{null}(A) = S$ then $\operatorname{row}(A) = S^{\perp}$ so we can take

$$\begin{bmatrix}1 & -1 & 0 & 0\\0 & 0 & 1 & -1\end{bmatrix}.$$

(d) (3 points) Does there exist a rank 3 matrix A such that $\operatorname{null}(A) = S$? If so, give an example. If not, why not?

Solution: No. By the rank-nullity theorem, $\operatorname{rank}(A) + \operatorname{null}(A) = 4$. Since $\dim S = 2$, the rank of A must be 2.

5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transform defined by the following properties:

- $T(0, 0, 1) = (0, 0, 0)$,
- If v is in the xy-plane, then v is reflected across the $x + y = 0$ plane.

There is a matrix A such that $T(x) = Ax$. The goal of this problem is to understand A.

(a) (3 points) Find a basis $\{u, v, w\}$ where the action of T is well-understood. Give also $T(u), T(v)$, and $T(w)$.

Solution:

$$u = (0, 0, 1), T(u) = (0, 0, 0)$$
$$v = (1, 1, 0), T(v) = (-1, -1, 0)$$
$$w = (1, -1, 0), T(w) = (1, -1, 0)$$

(b) (3 points) Find the eigenvalues of A and a basis for each eigenspace of A. (Think geometrically.)

Solution: Part (a) gives the answer.
$\lambda = 0$ is an eigenvalue with eigenspace spanned by u.
$\lambda = -1$ is an eigenvalue with eigenspace spanned by v.
$\lambda = 1$ is an eigenvalue with eigenspace spanned by w.

(c) (3 points) What is A? You may express it as product of matrices and their inverses.

Solution: Using the theory of diagonalization,

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}^{-1}$$

(d) (3 points) What is A^2? Give it explicitly as a single matrix. (Think geometrically.)

Solution: We can see that A^2 is projecting onto the xy-plane. So

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

6. Let A be the symmetric matrix defined as

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1}.$$

(a) (3 points) Find the eigenvalues of A and a basis for each eigenspace of A.

Solution: $\lambda = -1$ is an eigenvalue with $\{(1, 1, 1)\}$ as a basis for its eigenspace.
$\lambda = 2$ is an eigenvalue with $\{(-1, 0, 1), (-2, 1, 1)\}$ as a basis for its eigenspace.

(b) (3 points) Find a basis for each of the following subspaces.

- null(A)

Solution: Since 0 is not an eigenvalue, null(A) $= \{0\}$ with basis $\emptyset$.

- null($A - I$)

Solution: Since 1 is not an eigenvalue, null(A) $= \{0\}$ with basis $\emptyset$.

- null($A - 2I$).

Solution: We have that null($A - 2I$) $= E_2$ which has basis $\{(-1, 0, 1), (-2, 1, 1)\}$.

(c) (3 points) Find an orthogonal matrix Q and a diagonal matrix D such that $A = QDQ^{-1}$.

Solution: We use Gram-Schmidt to perform an orthogonal basis for each eigenspace.
An orthonormal basis for the eigenspace corresponding to $\lambda = -1$ is $\{(1/3, 1/3, 1/3)\}$.
An orthonormal basis for $\lambda = 2$ is $\{\frac{1}{\sqrt{2}}(-1, 0, 1), \sqrt{\frac{2}{3}}(-1/2, 1, -1/2)\}$.

(d) (3 points) Find all $k \in \mathbb{R}$ such that $A - kI_3$ is not invertible.

Solution: $k = -1, 2$.

7. Let $v = (2, 2, 1)$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x) = \text{proj}_v x$.

(a) (4 points) Find an orthogonal basis for $\mathbb{R}^3$ that contains v. (Hint: first find a basis for $\mathbb{R}^3$ that contains v.)

Solution: $\{(2, 2, 1), (1, 0, -2), (0, 1, -2)\}$.

(b) (4 points) There exists a matrix A such that $T(x) = Ax$. Find the eigenvalues of A and a basis for each eigenspace of A. (Hint: see part (a).)

Solution: The eigenspace corresponding to 1 is spanned by $(2, 2, 1)$.
The eigenspace corresponding to 0 is spanned by $(1, 0, -2), (0, 1, -2)$.

(c) (4 points) Let $e_1 = (1, 0, 0)$. Evaluate the following:

- Ae_1

Solution: This is $T(e_1) = \text{proj}_v e_1 = (4/9, 4/9, 2/9)$.

- $A^2 e_1$

Solution: Doing two projections is the same as one.

- $A^{100} e_1$

Solution: Doing one hundred projections is the same as one.