Row and column spaces

Definition: Let A be $n \times m$ matrix. Then

- The **row space**, denoted $row(A)$, of A is the subspace of $\mathbb{R}^m$ given by the span of the rows of A.
- The **column space**, denoted $col(A)$, of A is the subspace of $\mathbb{R}^n$ given by the span of the columns of A.

Theorem: Let A be a matrix and B an echelon form of A.

- The nonzero rows of B form a basis for $row(A)$.
- The columns of A corresponding to the pivot columns of B form a basis for $col(A)$.

Consequently, the dimension of the row space and the columns space of A are the same. We call this the rank of A, denoted $rank(A)$.

Example: Let A be

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & -1 & 2 & 1 \\ 5 & 0 & 1 & -1 \end{bmatrix}$$

Find a basis for the row space. Find a basis for the column space. Determine the rank of A.

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# We compute the rref of A and stare at it
A = matrix([[1,2,3,4],[3,-1,2,1],[5,0,1,-1]])
B = A.rref(); B
# The first 3 columns of A form a basis (so does the standard basis) for the column space
# The rows of B form a basis for the row space
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[      1      0      0 -13/28]
[      0      1      0    1/4]
[      0      0      1  37/28]
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Definition: The **nullity** of a matrix A, denoted $null(A)$, is the dimension of the solution space to $Ax = 0$.

Example: What is the nullity of the previous A? (It is 1).

Theorem: (Rank-Nullity Theorem) Let A be a $n \times m$ matrix. Then $rank(A) + nullity(A) = m$.

Linear transform perspective

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transform. Let A be the matrix so that $T(x) = Ax$. Then $range(T) = col(A)$ so we know that hte rank of A is the dimension of the range. We know that the nullity is the dimension of the kernel. So dimension of range + dimension of kernel is the dimension of the domain.