

5.1/5.2 - Determinants

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transform and A be the square matrix defined so that $T(x) = Ax$. Then the absolute value of the determinant of A , denoted $\det(A)$, measures the change in volume under T . This means that if S is a shape of volume V then $T(S) = \{T(s) : s \in \mathbb{R}^n\}$ has volume $|\det(A)|V$.

Definition: For any $n \in \mathbb{N}$, The **determinant** is the unique function from the set of square matrices of size n to the real numbers with the following 3 properties:

- $\det(I_n) = 1$,
- When viewing a square matrices as a list of n column vectors, the determinant is n -linear. This means $\det([a_1, \dots, ca_i + db_i, \dots, a_n]) = c \det([a_1, \dots, a_i, \dots, a_n]) + d \det([a_1, \dots, b_i, \dots, a_n])$.
- If there is a column of zero, then the determinant is zero.

The determinant of a linear transform $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the determinant of its associated matrix.

Properties:

- $\det(AB) = \det(A) \det(B)$,
- $\det(A^t) = \det(A)$,
- If A is upper (or lower) triangular, then $\det(A)$ is the product of the diagonal,
- $\det(cA) = c^n \det(A)$,
- If A has positive nullity, then $\det(A) = 0$.

Computation: There is a thing called cofactor expansion. It is terrible but we will learn it. In practice, another method is used like LU -decomposition. There LU decomposition of a matrix A is $A = LU$ where L is lower triangular and U is upper triangular. Then $\det(A) = \det(L) \det(U)$, where $\det(L)$ and $\det(U)$ is the product of the diagonal. Give $n = 2$ and $n = 3$ shortcuts and the general cofactor formula. Note that you can expand along any row or column.

Notation: Often you denote the determinant of a matrix by replacing the square brackets by straight lines.

Examples: Compute the determinant in the following cases:

- $T(x, y) = (x + y, y)$
- $T(x, y) = (3x + y, -y)$. Also what is the determinant of T^{-1} here? (demonstrate that $\det(A^{-1}) = \det(A)^{-1}$ here)
- Here is a 3×3 example
- Here is a 4×4 example that I won't actually work out fully

Theorem: Let $S = \{a_1, \dots, a_n\}$ be a set of vectors in \mathbb{R}^n , let $A = [a_1 \dots a_n]$ and $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be given by $T(x) = Ax$. Then the following are equivalent:

- S spans \mathbb{R}^n ,

- S is linearly independent,
- S is a basis for \mathbb{R}^n ,
- $Ax = b$ has a unique solution for every $b \in \mathbb{R}^n$,
- T is onto,
- T is one-to-one,
- $\ker T = \{0\}$,
- $\text{range}(T) = \mathbb{R}^n$,
- $\text{col}(A) = \mathbb{R}^n$,
- $\text{row}(A) = \mathbb{R}^n$,
- $\text{rank}(A) = n$,
- $\text{nullity}(A) = 0$,
- Any echelon form of A has no zero entries on the diagonal,
- The reduced echelon form of A is the identity matrix,
- $\det(A) \neq 0$,

tldr: Determinant tells you about change of volume. It is a test of invertibility.