Eigenvalues and Eigenspaces

- Watch 13 and 14 of 3blue1brown

(insert picture of what a eigenvector is)

Definition: Let A be a $n \times n$ matrix. Then a nonzero vector u is an *eigenvector* if there exists a scalar λ such that $Au = \lambda u$. The scalar λ here is called the *eigenvalue*. Here u is an eigenvector associated to λ.

Examples:

- What are the eigenvalues and eigenvalues of a diagonal matrix?
- What are the eigenvalues and eigenvectors of problem 3 on the midterm?
- What are the eigenvalues and eigenvectors of reflection across a plane?
- Let $A = [[3, 5], [4, 2]]$. Determine if each of the following is an eigenvector for A. $u_1 = (5, 4), u_2 = (4, -1), u_3 = (-1, 1)$.

Theorem A square matrix is invertible if and only if 0 is not a eigenvalue.

Theorem/Definition: Let A be a $n \times n$ matrix with eigenvalue λ. Then the set of all eigenvectors associated to λ along with 0 forms a subspace, called the *eigenspace*, of $\mathbb{R}^n$. This is also the null space of $A - \lambda I$.

Theorem/Definition: Let A be an $n \times n$ matrix. Then λ is an eigenvalue if and only if $\det(A - \lambda I) = 0$. The polynomial $\det(A - \lambda I)$ is called the *charateristic polynomial* of A. The *multiplicity* of a eigenvalue is its multiplicity in the charateristic polynomial.

Example: Find the eigenvalues and a basis for each eigenspace for $A = [[0, 2, -1], [1, -1, 0], [1, -2, 0]]$.

It turns out that $\det(A - \lambda I)$ is $-\lambda^3 - \lambda^2 + \lambda + 1 = -(\lambda - 1)(\lambda + 1)^2$.

So we are just finding the basis for the nullspaces of $A - I$ and $A + I$ which we can do with row reductions.

Theorem: Let A be a square matrix with eigenvalue λ. Then the dimension of the associated eigenspace is less than or equal to the multiplicty of λ.