# October 9

## Announcements

- Section 2.1, 2.2 due this Thursday
- Section 2.3, 3.1 due next Thursday
- Midterm next Wednesday in class
  - 1.1 - 3.1 (maybe 3.2)
- Worksheet 1 solutions has been posted
- Worksheet 2 has been posted, due this Friday
- Wednesday office hours to be held in classroom?

## Homogenous Systems

Let $A$ be a matrix. Then $A(x+y) = Ax + Ax$ and $A(x-y) = Ax - Ay$.

**Example:** Find a general solution for the linear system $

$$\begin{aligned} 2x_1 - 6x_2 - x_3 + 8x_4 &= 7 &(1)\\ x_1 - 3x_2 - x_3 + 6x_4 &= 6 &(2)\\ -x_1 + 3x_2 - x_3 + 2x_4 &= 4. &(3) \end{aligned}$$

$ Using row reduction, we see that a general solution is of the form $x = (1, 0, -5, 0) + s_1(3, 1, 0, 0) + s_2(-2, 0, 4, 1)$.

The solution to the homogenous system is $x = s_1(3, 1, 0, 0) + s_2(-2, 0, 4, 1)$.

Let $x_p$ be a particular solution $Ax = b$. Then solutions have the form $x_g = x_p + x_h$, where $x_p$ is a particular solution and $x_h$ is the general solution to the homogenous equations.

## Linear Indepedence and Span

**Theorem:** Let $A = [a_i]$ and $b$ be a vector in $\mathbb{R}^n$. Then the following are equivalent (if one is true then they are all true, if one is false then they are all false).

- The set $\{a_1, \ldots, a_m\}$ are linearly independent.
- The vector equation $x_1a_1 + x_2a_2 + \ldots + x_ma_m = b$ has at most one solution.
- The linear system $[a_1 \ a_2 \ \ldots \ a_m | b]$ has at most one solution.
- The equation $Ax = b$ has at most 1 solution.

**Example:** Consider the vectors $a_1 = (1, 7, -2)$, $a_2 = (3, 0, 1)$, and $a_3 = (5, 2, 6)$. Set $A = [a_i]$. Show that the columns of $A$ are linearly independent and that $Ax = b$ has a unique solution for every $b$ in $\mathbb{R}^3$.

**Example:** Let $u_1 = (1, -1, 2), u_2 = (2, -1, 2), u_3 = (-2, 5, -10), u_4 = (3, -4, 8)$.

The associated matrix has reduced echelon form:

$$\begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Is $\{u_1, \ldots, u_4\}$ linearly independent? Can we write $u_1$ as a linear combination of $u_2, \ldots, u_4$?

If a set of vectors is not linearly indepedent, can every vector be written as a linear combination of the other vectors? In other words, is every vector in the span of the other vectors?

## Section 3.1 Linear Transformations

We can write linear equations as $Ax = b$. We can think of it as $A$ sending $x$ to $b$.

**Definition:** A function $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation if for all vectors $u, v \in \mathbb{R}^m$ and all scalars $r$, we have

- T(u+v) = T(u) + T(v)
- T(ru) = rT(u).

**Examples:**

- What are some examples of functions that aren't linear transforms? quadratic, ax+b
- Consider the function given by $T(x_1, x_2) = (3x_1 - x_2, 2x_1 + 5x_2)$. What is $T(1, 2)$? Show that this is a linear transformation. Is it associated to a matrix?
- Projections are linear transforms.
- Let $A$ be some matrix. Then $T(x) = Ax$ is a linear transform. Make up some example in class.

A matrix, $A$, is said to be an $n \times m$ matrix if it has $n$ rows and $m$ columns. If $m = n$, then $A$ is a sqaure matrix.

**Theorem:** Let $A$ be an $n \times m$ matrix, and define $T(x) = Ax$. Then $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transform. Moreover, all linear transform are of this form.

**Example:** Consider the linear transform with matrix

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \end{bmatrix}.$$

Is $(3, 4)$ in the range of $A$?

**Theorem:** Let $A = [a_1 \ a_2 \ \ldots \ a_m$ be a $n \times m$ matrix, and let $T : \mathbb{R}^m \to \mathbb{R}^n$ with $T(x) = Ax$ be a linear transformation. Then

- A vector $w$ is in the range of $T$ if and only if $Ax = w$ is a consistent linear system.

- The range of $T$ is the span of the columns (this is also called the column space).

If time, talk about 1-1 and onto