October 11

Announcements

- Section 2.1, 2.2 due tomorrow
- Section 2.3, 3.1 due next Thursday
- Worksheet 2 due Friday
- Midterm next week
- Worksheet 3 will be posted on Friday, it'll have some practice exam problems

3.1 Linear transformation

One-to-one and Onto linear transformation

Definition: Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Then

- T is *one-to-one* if for every vector $w \in \mathbb{R}^n$, there exists at most one vector $v \in \mathbb{R}^m$ such that $T(v) = w$.
- T is *onto* if for every vector $w \in \mathbb{R}^n$, there is exists at least one vector $v \in \mathbb{R}^n$ such that $T(v) = w$.

A linear transformation T is one-to-one if $T(u) = T(v)$ implies $u = v$. In other words, if $u \neq v$, then $T(u) \neq T(v)$. (Two-to-two!)

Talk about the general idea of one-to-one and onto.

Theorem: Let T be a linear transformation T is one-to-one if $T(u) = 0$ implies $u = 0$.

Example: Let T be the linear transformation defined by $T(x) = Ax$, where

$$\begin{bmatrix} 4 & -1 \\ -2 & 2 \\ 0 & 3 \end{bmatrix}$$

Is T one-to-one? Onto?

Let T be the linear transformation defined by $T(x) = Ax$, where

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{bmatrix}$$

Is T one-to-one? Onto?

Theorem: Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Let A be the matrix so that $T(x) = Ax$. Then

- T is one-to-one if the columns of A are linearly independent.
- T is onto if the columns of A span $\mathbb{R}^n$

In particular, the dimension of A can sometimes implies that T cannot be one-to-one and onto.

Theorem: Let $S = \{a_1, \ldots, a_n\}$ with $a_i \in \mathbb{R}^n$, $A = [a_i]$, and $T(x) = Ax$. (So A is square). Then the following are equivalent:

- S spans $\mathbb{R}^n$
- S is linearly independent
- $Ax = b$ has a unique solution for all $b \in \mathbb{R}^n$
- $T(xs) = b$ has a unique solution for all $b \in \mathbb{R}^n$
- T is onto
- T is one-to-one.

Geometry of linear transformations from R^2 to R^2

Lines go to lines (or points)! Why? $T((1-s)u + sv) = (1-s)T(u) + sT(v)$.

The columns of the matrix tells you where the standard basis goes.

Let's see what happens to the square $\{(x, y) : 0 \leq x, y \leq 1\}$ under the following transforms

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$