October 11

Announcements

- Section 2.3, 3.1 due next Thursday
- Write down name if you did worksheet 2
- Midterm next week
- Worksheet 3 will be posted tonight, it'll have some practice exam problems

3.1 Linear transformation

Theorem: Let $S = \{a_1, \ldots, a_n\}$ with $a_i \in \mathbb{R}^n$, $A = [a_i]$, and $T(x) = Ax$. (So A is square). Then the following are equivalent:

- S spans $\mathbb{R}^n$
- S is linearly independent
- $Ax = b$ has a unique solution for all $b \in \mathbb{R}^n$
- $T(xs) = b$ has a unique solution for all $b \in \mathbb{R}^n$
- T is onto
- T is one-to-one.

Geometry of linear transformations from R^2 to R^2

Lines go to lines (or points)! Why? $T((1-s)u + sv) = (1-s)T(u) + sT(v)$.

The columns of the matrix tells you where the standard basis goes. Once you know this, you should know everything.

Let's see what happens to the square $\{(x, y) : 0 \leq x, y \leq 1\}$ under the following transforms

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Piecing things together

Theorem: Let $S = \{v_1, \ldots, v_n\}$. Let A be the matrix with the elements of S as columns. Let B be an echelon matrix equivalent to A. Let T be a linear transform with $T(x) = Ax$. Then the following are equivalent

- The set S is linearly independent.
- The linear equation $x_1 v_1 + \ldots + x_n v_n = 0$ has only the trivial solution.

- Every columns of B has a pivot. (computationally useful)
- For any $b \in \mathbb{R}^n$, the equation $x_1v_1 + \ldots + x_nv_n = b$ has a unique solution.
- The homogenous equation $Ax = 0$ has only the trivial solution.
- For any $b \in \mathbb{R}^n$, the equation $Ax = b$ has at most one solution.
- For any $b \in \mathbb{R}^n$, b can be expressed as a linear combination of elements in S in at most one way.
- The zero vector can be expressed as a linear combination of elements in S in only one way.
- T is a one-to-one linear transformation.
- The only solution to $T(x) = 0$ is $x = 0$. If $T(x) = 0$, then $x = 0$.
- There is at most one solution to $T(x) = b$.

Theorem: Let $S = \{v_1, \ldots, v_n\}$ be a set of vectors in $\mathbb{R}^m$. Let A be the matrix with the elements of S as columns. Let B be an echelon matrix equivalent to A. Let T be a linear transform with $T(x) = Ax$. Then the following are equivalent

- The set S spans $\mathbb{R}^m$.
- The linear equation $x_1v_1 + \ldots + x_nv_n = b$ always has a solution.
- Every row of B has a pivot. (computationally useful)
- For any $b \in \mathbb{R}^n$, the equation $Ax = b$ has at least one solution.
- For any $b \in \mathbb{R}^n$, b can be expressed as a linear combination of elements in S in at least one way.
- T is a onto linear transformation.
- There is always a solution to $T(x) = b$.

Examples:

Kristin DeVleming exam: Let $u_1 = (4, 4, 2)$ and $u_2 = (8, 5, -3)$. Let $v = (26, 17, -8)$. Write v as a linear combination of u_1, u_2. Write a vector w that is not in the span of u_1, u_2.

Josh Swanson exam: Are the following sets spanning?

- $\{(1, 2, 3), (-1, -1, 2), (-1, 0, 7)\}$
- $\{(1, -1, 1), (0, 1, 2), (-2, 0, 2), (1, 3, 1)\}$.