

## Worksheet 2

Due 10/13

1. We know how to obtain the general solution from a linear system. Let's try to reverse it. Find a linear system whose general solution is

$$(x_1, x_2, x_3, x_4) = (1, 2, 3, 4) + s_1(5, 6, 7, 8) + s_2(9, 0, 1, 2).$$

**ANSWER:** This question turned out to be easier and harder than I thought.

The easier approach is to write out the 4 equations, solve for  $s_1$  and to get rid of it, then do the same for  $s_2$ .

Here's the harder approach I had in mind. We have 2 free variables. I will choose  $x_2$  and  $x_3$  to be the free variables. The goal is to manipulate the form of the general solution until it looks like

$$(x_1, x_2, x_3, x_4) = (X, 0, 0, X) + s_1(X, 1, 0, X) + s_2(X, 0, 1, X),$$

where  $X$  is any number.

Let  $u = (1, 2, 3, 4)$ ,  $v = (5, 6, 7, 8)$ ,  $w = (9, 0, 1, 2)$ . The general solution is equivalent to  $\{u + x : x \in \text{span}(v, w)\}$ . We have some freedom. We can replace  $u$  with any particular solution and we can replace  $v, w$  with any other 2 vectors with the same span.

Let  $v_2 = (v - 7w)/6$ . Then  $v_2$  and  $w$  has the same span as  $v, w$  and  $v_2 = (-29/3, 1, 0, -1)$ .

Let  $u_2 = u - 2v_2 - 3w$ . So  $u_2 = (-20/3, 0, 0, 0)$ .

We now have that

$$(x_1, x_2, x_3, x_4) = (-20/3, 0, 0, 0) + s_1(-29/3, 1, 0, -1) + s_2(9, 0, 1, 2)$$

is of the desired form. By setting  $x_2 = s_1$  and  $x_3 = s_2$ , we see that  $x_1 = -20/3 - 29/3x_2$  and  $x_4 = -x_2 + 2x_3$ .

2. Suppose  $A$  is a matrix. Let  $v, w$  be distinct (meaning  $x \neq y$ ) vectors that solve  $Ax = 0$  so  $Av = 0$  and  $Aw = 0$  (0 here of course means the zero vector!). Let  $L$  be the line that passes through  $v$  and  $w$ . If  $u$  is on  $L$ , then  $Au = 0$ . Why? This exercise suggests that solution spaces are convex.

**ANSWER:**

If  $u$  is on the line that passes through  $v$  and  $w$ , then  $u$  is of the form  $sv + (1-s)w$ . Then  $A(u) = sA(v) + (1-s)A(w) = s0 + (1-s)0 = 0$ .

3. Let  $z_1, z_2 \in \mathbb{R}$  and let  $S = \{(1, z_1, z_2), (2, 1, 0), (1, 0, -1)\}$ .
  - Find some values for  $z_1$  and  $z_2$  such that  $S$  spans  $\mathbb{R}^3$ .
  - Find some values for  $z_1$  and  $z_2$  such that  $S$  does not span  $\mathbb{R}^3$ .

- Find all values for  $z_1$  and  $z_2$  such that  $S$  spans  $\mathbb{R}^3$ . (In the process of solving this problem, some of you will be tempted to divide by zero. Resist that temptation.)

**ANSWER:** The problem becomes much easier once you reorder the vectors. This does not affect linear independence! The matrix use to determined linear independence is

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & z_1 \\ -1 & 0 & z_2 \end{bmatrix}$$

This matrix is much easier to use than the order the vectors were presented in. The nasty  $(1, z_1, z_2)$  is in the last column and the topright entry is a 1. This matrix is equivalent to

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & z_1 \\ 0 & 2 & 1 + z_2 \end{bmatrix}$$

by adding the first row to the last. We can then see that this matrix is has a pivot in each column whenever  $(1, z_1)$  is not parallel to  $(2, 1 + z_2)$ . So the vectors are linearly independent whenever  $2z_1 \neq 1 + z_2$ .

4. Consider the following linear system that came from the book and the lecture.

$$2x_1 - 6x_2 - x_3 + 8x_4 = 0 \quad (1)$$

$$x_1 - 3x_2 - x_3 + 6x_4 = 0 \quad (2)$$

$$-x_1 + 3x_2 - x_3 + 2x_4 = 0. \quad (3)$$

Using row reduction, we see that a general solution is of the form  $x = s_1(3, 1, 0, 0) + s_2(-2, 0, 4, 1)$ . Let  $v_1 = (2, 1, -1), v_2 = (-6, -3, 3), v_3 = (-1, -1, 2), v_4 = (8, 6, 2)$ .

- Is  $\{v_1, v_2, v_3, v_4\}$  is linearly independent set? The answer should be no.
- Express  $v_1$  as a linear combination of  $v_2, v_3, v_4$ .
- Express  $v_2$  as a linear combination of  $v_1, v_3, v_4$ .
- Express  $v_3$  as a linear combination of  $v_1, v_2, v_4$ .
- Express  $v_4$  as a linear combination of  $v_1, v_2, v_3$ .

**ANSWER:**

By setting  $s_1 = 1$  and  $s_2 = 2$ , we obtain a nontrivial solution to the system which implies  $v_1 + v_2 + 4v_3 + v_4 = 0$ . By solving for  $v_1$  here, we can write  $v_1$  as a linear combination of  $v_2, v_3, v_4$ . Same for the others.

5. Suppose  $\{v_1, v_2, v_3\}$  is a linearly dependent set. Is it always the case that we can write  $v_1$  as a linear combination of  $v_2$  and  $v_3$ ? If not, come up with a counterexample.

**ANSWER:**

No. Take  $v_1 = (1, 0)$ ,  $v_2 = (0, 1)$ ,  $v_3 = (0, 2)$ .

6. Come up with a inconsistent linear system whose associated homogenous linear system is consistent.

**ANSWER:**

A homogeneous linear system is always consistent so any inconsistent linear system is an example.