8/16

Taylor series

L'hospital rule

If $f(0) = g(0) = 0$ then

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)}$$

if it exists.

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \overset{(\text{since } f(0)=0)}{=} \sum_{k=1}^{\infty} a_k x^k$$

$$= \sum_{l=0}^{\infty} a_{l+1} x^{l+1} \qquad (l = k-1)$$

$$= x \cdot \sum_{l=0}^{\infty} a_{l+1} x^{l} .$$

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{x \cdot \sum_{i=0}^{\infty} a_{i+1} x^i}{x \cdot \sum_{j=0}^{\infty} b_{j+1} x^j} = \frac{a_1}{b_1} = \frac{f'(0)}{g'(0)} .$$

~~The order of vanishing of $f(x) = \sum_{k=0}^{\infty} a_k x^k$ is the first k such that $a_k \neq 0$~~

→ pretend I didn't cross that out.

It's not important for this class though.

Multiplying/dividing by a power of x.

$$f(x) = \frac{e^x - 1 - x}{x^2}.$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \sum_{k=2}^{\infty} \frac{x^k}{k!}$$

$$\Rightarrow \quad e^x - 1 - x = \sum_{k=2}^{\infty} \frac{x^k}{k!}$$

$$\Rightarrow \quad \frac{e^x - 1 - x}{x^2} = \sum_{k=2}^{\infty} \frac{x^{k-2}}{k!}$$

$$= \sum_{\ell=0}^{\infty} \frac{x^\ell}{(\ell+2)!} \qquad (\ell = k-2)$$

Making it look like $\frac{1}{1-x}$.

~~Diff/Integration~~.

based at b=0.

$$\frac{1}{5 - 3x^2} = \frac{1}{5(1 - \frac{3}{5}x^2)} = \frac{1}{5}\sum_{k=0}^{\infty} \left(-\frac{3}{5}x^2\right)^k$$

pull out the constant factor to make it a 1.

$$= \sum_{k=0}^{\infty} \frac{(-3)^k}{5^{k+1}} x^{2k}$$

$$\left|\frac{-3}{5}x^2\right| < 1$$

$$\Rightarrow \quad |x| < \sqrt{5/3}$$

Diff/ Integration

based at 0

$$\ln(2+x) = \cancel{\int_0^x \frac{1}{2+t}\,dt} \int \frac{1}{2+x}\,dx$$

$$= \int \frac{1}{2(1-(-\frac{x}{2}))}\,dx$$

$$= \int \frac{1}{2}\sum_{k=0}^{\infty}\left(-\frac{1}{2}\right)^k x^k\,dx$$

$$= \int \sum_{k=0}^{\infty}\frac{(-1)^k}{2^{k+1}}x^k\,dx$$

$$= \sum_{k=1}^{\infty}\frac{(-1)}{2^{k+1}\cdot(k+1)}\cdot x^{k+1} + C$$

Plug in $x=0$ to determine C.

$$\ln(2) = C$$

$$\ln(2) + \sum_{k=1}^{\infty}\frac{(-1)}{2^{k+1}\cdot(k+1)}x^{k+1}$$

radius $\left|\frac{-x}{2}\right| < 1 \Rightarrow |x| < 2$.

Scanned by CamScanner

$f(x) = 2xe^{x^2}$

2 ways.

$f(x) = \frac{d}{dx}\left(e^{x^2}\right)$

$= \frac{d}{dx} \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}$

$= \sum_{k=1}^{\infty} \frac{2k \cdot x^{2k-1}}{k!}$

$= \sum_{k=1}^{\infty} \frac{2x^{2k-1}}{(k-1)!}$

$(\ell = k-1)$

$= \sum_{\ell=0}^{\infty} \frac{2x^{2\ell+1}}{\ell!}$

$f(x) = 2x \cdot \sum_{k=0}^{\infty} \frac{x^k}{k!}$

$= \vdots$

Scanned by CamScanner