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13.4 Motion in Space : Velocity and Acceleration

In this lecture, assume r is a vector function that represents position.

velocity$\overset{\text{at time } t}{\downarrow}$: $r'(t)$

acceleration$\overset{\text{at time } t}{\downarrow}$: $r''(t) = v'(t)$

displacement from t_0 to t_1 : $r(t_1) - r(t_0)$

speed at time t : $|v(t)| = |r'(t)|$

distance traveled from t_0 to t_1 : $\int_{t_0}^{t_1} |v(t)|\,dt$ ⇙ arclength!

Ex

Find the velocity, acceleration, and the speed of a particle with position $r(t) = \langle t^2, e^t, te^t \rangle$.

$$v(t) = r'(t) = \langle 2t, e^t, e^t + te^t \rangle$$

$$a(t) = v'(t) = r''(t) = \langle 2, e^t, 2e^t + te^t \rangle$$

$$s(t) = |v(t)| = |\langle 2t, e^t, e^t + te^t \rangle|$$

$$= \sqrt{4t^2 + e^{2t} + (e^t + te^t)^2}$$

Find the displacement from $t = 0$ to $t = 1$.

$$r(1) - r(0) = \langle 1, e, e \rangle - \langle 0, 1, 0 \rangle$$

$$= \langle 1, e-1, e \rangle.$$

Find the distance travelled from $t=0$ to $t=1$.

$$\int_0^1 \sqrt{4t^2 + e^{2t} + (e^t + te^t)^2}\, dt.$$

ex

A particle has ~~acceleration~~ initial position ~~@~~

~~v(0)~~ $r(0) = \langle 1, a, 0 \rangle$, initial velocity is

$v(0) = \langle 1, -1, 1 \rangle$. Its acceleration is

$a(t) \langle 4t, 6t, e^t \rangle$.

Find the ~~vel~~ velocity and position at time t.

Since, $a(t) = v'(t)$, we have

$$v(t) = \int a(t)\, dt$$

$$= \int \langle 4t, 6t, e^t \rangle dt$$

$$= \langle 2t^2, 3t^2, e^t \rangle + C,$$

~~$v(0) = \langle 1, -1, 1 \rangle$~~

$$\langle 1, -1, 1 \rangle = v(0)$$

$$= \langle 0, 0, 1 \rangle + C$$

$$\Rightarrow \quad C = \langle 1, -1, 0 \rangle$$

$$\Rightarrow \quad v(t) = \langle 2t^2 + 1, 3t^2 - 1, e^t \rangle$$

$$r(t) = \int v(t)\,dt$$

$$= \int \langle 2t+1,\ 3t^2-1,\ e^t \rangle\,dt$$

$$= \langle t^2+t,\ t^3-t,\ e^t \rangle + C$$

$$\langle 1,0,0 \rangle = r(0) = \langle 0,0,1 \rangle + C$$

$$\Rightarrow C = \langle 1,0,-1 \rangle$$

$$\Rightarrow r(t) = \langle t^2+t+1,\ t^3-t,\ e^t-1 \rangle.$$

Tangential and Normal components of Acceleration.

velocity and acceleration ~~velocity~~ occurs in the osculating plane.

Let $s = |v|$ be the speed. Then

$$v = s \cdot T.$$

By differentiating both sides,

$$a = s' \cdot T + s \cdot T'$$

But $K = \frac{|T'|}{|r'|} = \frac{|T'|}{s}$ so $|T'| = ks$. The unit normal vector is $N = \frac{T'}{|T'|}$ so $T' = |T'| \cdot N$

$$a = s' \cdot T + ks^2 \cdot N.$$

Let $a_T = v'$ and $a_N = kv^2$. Then

$$a = a_T \cdot T + a_N \cdot N.$$

We also have

$$a_T = \text{comp}_T a$$

and $a_N = \text{comp}_N a$.

These are the tangential and normal components of acceleration. ~~Therefore~~ and we have

$$a_T = \frac{r'(t) \cdot r''(t)}{|r'(t)|}$$

and $a_N = \cancel{\frac{|r'(t) \times r''(t)|}{|r'(t)|}}$

$$kv^2 = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} \cdot |r'(t)|^2$$

$$= \frac{|r'(t) \times r''(t)|}{|r'(t)|}.$$

Find a_N and a_T for $r(t) = \langle t, t^2, t^3 \rangle$

$$r'(t) = \langle 1, 2t, 3t^2 \rangle$$

$$r''(t) = \langle 0, 2, 6t \rangle$$

$$a_T = \frac{4t + 18t^3}{\sqrt{1+4t^2+9t^4}} \quad \Big/ \quad a_N = \frac{|\langle 12t^2 - 6t^2, -6t, 2 \rangle|}{\sqrt{1+4t^2+9t^4}}.$$

Ex

A projectile is fired with angle of elevation α and initial ~~velocity v_0~~ speed $s_0 = |v_0|$. What value of α maximizes the distance (horizontal distance traveled)?

The acceleration due to gravity is

$$a = -g j.$$

Since $v'(t) = a$, we have

$$v(t) = -gt\, j + C,$$

where $C = v(0) = v_0$. Therefore,

$$r'(t) = v(t) = -gt j + v_0.$$

Integrating again,

$$r(t) = -\frac{1}{2} g t^2 j + t v_0 + D$$

But $r(0) = D = 0$, so

$$r(t) = -\frac{1}{2} g t^2 j + t v_0.$$

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V_0

If v_0 is the initial velocity, then

$$v_0 = s_0 \cos\alpha\, i + s_0 \sin\alpha\, j.$$

so then

$$r(t) = (s_0 \cos\alpha)\, t\, i + \left[(s_0 \sin\alpha) t - \tfrac{1}{2} g t^2\right] j.$$

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