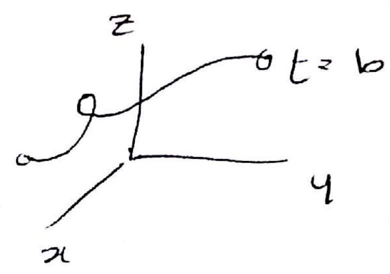
§13.3 Arc Length and Curvature:



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Last Week we mentioned:

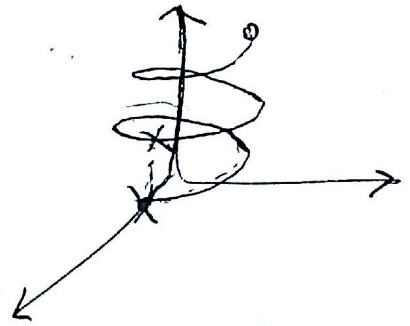
$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

Notice we may write this formula more compactly if we let $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ be the position along the curve, then

$$L = \int_a^b |\vec{r}\,'(t)|\, dt.$$

Note: Computation of length is independent of param.

Eg 1. Find the length of one "period" of the helix $\vec{r}(t) = \cos t\, \hat{i} + \sin t\, \hat{j} + t\hat{k}$.



[e.g. $(1,0,0)$ to $(1,0,2\pi)$.]

This is where things get weird...

If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is a param that traverses a curve once, then

First, we define the arclength function from $t = a$ by $s(t) = \int_a^t |\vec{r}\,'(u)|\, du$

Now use the, oft forgot, second FTC which states

$$F(t) = \int_a^t f(u)\,du$$
$$\Rightarrow F'(t) = f(t).$$

Thus, $\frac{ds}{dt} = |\vec{r}\,'(t)|$.

So, sometimes, it is useful to parameterize a curve w.r.t. arclength as it arises naturally from the shape of the curve, and is a "canonical" parameterization. i.e. there is only one way to param w.r.t. arclength. (two if you count orientation).

Eg 2. Reparam the helix $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\hat{k}$ w.r.t. arclength measured from $(1,0,0)$ in the direction of increasing t.

~~Since $(1,0,0) = t=0$ we know~~

Notice $\frac{ds}{dt} = |\vec{r}\,'(t)| = \sqrt{\cos^2 t + \sin^2 t + 1^2} = \sqrt{2}$.

Hence, $s = s(t) = \int_0^t |\vec{r}\,'(u)|\,du = \int_0^t \sqrt{2}\,du = \sqrt{2}\,t$.

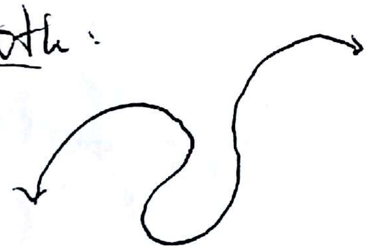
Thus, $s = \sqrt{2}t \Rightarrow t = s/\sqrt{2}$. $\rightsquigarrow$

Curvature

We call a parameterization smooth on an interval I if $\vec{r}\,'$ is continuous and $\vec{r}\,'(t) \neq \vec{0}$ on I.

We then call a curve smooth if it has a smooth parameterization. Note: Having a smooth parameterization does not imply that all parameterizations are smooth.

Smooth:

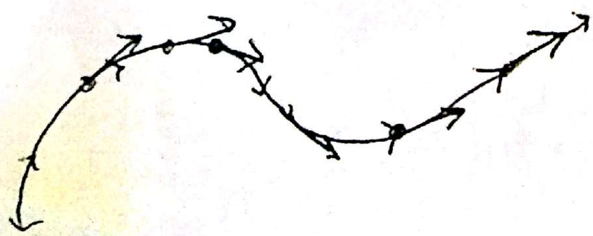


Non-smooth:

cusp, corners etc.

If C is a smooth curve with param given by $\vec{r}(t)$, then we define the unit tangent vector $\hat{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$.

Such a vector points in the direction of the curve.



Near sharper turns, the unit tangent changes quick

As a measure of "how fast the curve is changing direction" we define

$$K = \left|\frac{d\vec{T}}{ds}\right|.$$

Kappa

Note we may rewrite curvature as:

$$K = \left|\frac{d\vec{T}/dt}{ds/dt}\right| = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}.$$

Eg. 3.

Show that the curvature of a circle of radius a is $1/a$ at every point.

Take $\vec{r}(t) = a\cos t\,\hat{i} + a\sin t\,\hat{j}$

; Calc $|\vec{T}'(t)|$ and $|\vec{r}'(t)|$.

This shows that small circles have large curvature and large circles have small curvature.

Pickup Quiz from Nico