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## Linear combinations and span

**Definition:** If  $u_1, u_2, \dots, u_m$  are vectors and

$c_1, c_2, \dots, c_m$  are scalars, then

$$c_1 u_1 + c_2 u_2 + \dots + c_m u_m$$

is a *linear combination* of  $u_1, \dots, u_m$ . Note that the constants can be negative or zero.

**Definition:** Let  $S = \{u_1, u_2, \dots, u_m\}$  be a set of vectors. Then the span of  $S$ ,  $\text{span} S$ , is the set of all linear combinations of  $u_1, u_2, \dots, u_m$ .

What vectors in  $\mathbb{R}^2$  are a linear combination of  $(1, 0)$  and  $(0, 1)$ ? In other words, what vectors are in the span of  $(1, 0)$  and  $(0, 1)$ ?

What vectors in  $\mathbb{R}^2$  are a linear combination of  $(1, 2)$  and  $(0, 1)$ ? Talk about lines and averages here.

Is  $(3, 4)$  a linear combination of  $(1, 2)$  and  $(0, 1)$ ? In other words, is  $(3, 4)$  in the span of  $(1, 2)$  and  $(0, 1)$ ? In other words, does there exist  $x_1, x_2 \in \mathbb{R}$  such that  $x_1(1, 2) + x_2(0, 1) = (3, 4)$ ? In other words, system of equations!

Every system of equation can be interpreted in this way.

**Theorem:** Let  $u_1, \dots, u_m$  and  $v$  be vectors in  $\mathbb{R}^n$ . Then  $v \in \text{span}(\{u_1, \dots, u_m\})$  if and only if the linear system with augmented matrix  $[u_1 \ u_2 \ \dots \ u_m | v]$  has a solution.

The solution space can be expressed as a linear combination.

**Theorem:** Let  $u_1, u_2, \dots, u_m$  be vectors in  $\mathbb{R}^n$ . If  $u \in \text{span}(\{u_1, \dots, u_m\})$ , then  $\text{span}(\{u_1, \dots, u_m\}) = \text{span}(\{u_1, \dots, u_m, u\})$ .

When does a set of vectors span  $\mathbb{R}^n$ ?

**Theorem:** Let  $u_1, u_2, \dots, u_m$  be vectors in  $\mathbb{R}^n$ . Let  $A = [u_1 \ u_2 \ \dots \ u_m]$  and  $B \sim A$ , where  $B$  is in echelon form. Then  $\text{span}(\{u_1, \dots, u_m\}) = \mathbb{R}^n$  if and only if  $B$  has a pivot position in every row.

We can write linear systems as  $Ax = b$ .