Jan 17

Announcements

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Linear Independence

Let v, w be any vectors in $\mathbb{R}^n$. How does the span of $\{v, w\}$ compare to the span of to the span of $\{v, w, 2v + 3w\}$?

Consider the matrix this 3x3 matrix where the last row is the sum of the first two. What's the echelon form?

In these 2 examples, there were some redudant information.

Definition: Let $S = \{u_1, u_2, \ldots, u_m\}$ be a set of vectors in $\mathbb{R}^n$. We say that S is *linearly independent* if the only if the only solution to the vector equation

$$x_1u_1 + x_2u_2 + \ldots + x_mu_m = 0$$

is the trivial solution - $x_1 = x_2 = \ldots = x_m = 0$. If a set if not linearly indepedent then it is linearly dependent.

A set is linearly dependent iff some vector is in the span of the others. A set is linearly independent iff no vector is in the span of the others.

Any set containing the zero vector is linearly dependent.

Example: Is the set $\{(16, 2, 8), (22, 4, 4), (18, 0, 4), (18, 2, 6)\}$ linearly independent?

work out example in class using a linear system

Let $S = \{u_1, \ldots, u_m\}$ be a set of vectors in $\mathbb{R}^n$ and $A = [u_1\ u_2\ \ldots\ u_m]$ be the matrix formed by these vectors. Then S is linearly independent if and only if the only solution is the trivial solution.

Theorem: Let $S = \{u_1, \ldots, u_m\}$ be a set of vectors in $\mathbb{R}^n$. Suppose

$$A = [u_1\ u_2\ \ldots\ u_m] \sim B,$$

where B is in echelon form. Then * S spans $\mathbb{R}^n$ exactly when B has a pivot position in every row * S is linearly independent exactly when B has a pivot position in every column.

A set with fewer than n vectors will never span $\mathbb{R}^n$. A set with more than n vectors will never be linearly independent.

Homogenous Systems

Let A be a matrix. Then $A(x + y) = Ax + Ax$ and $A(x - y) = Ax - Ay$.

Example: Find a general solution for the linear system

$$2x_1 - 6x_2 - x_3 + 8x_4 = 7 \quad (1)$$
$$x_1 - 3x_2 - x_3 + 6x_4 = 6 \quad (2)$$
$$-x_1 + 3x_2 - x_3 + 2x_4 = 4. \quad (3)$$

Using row reduction, we see that a general solution is of the form $x = (1, 0, -5, 0) + s_1(3, 1, 0, 0) + s_2(-2, 0, 4, 1)$.

The solution to the homogenous system is $x = s_1(3, 1, 0, 0) + s_2(-2, 0, 4, 1)$.

Let x_p be a particular solution $Ax = b$. Then solutions have the form $x_g = x_p + x_h$, where x_p is a particular solution and x_h is the general solution to the homogenous equations.

Theorem: Let $A = [a_i]$ and b be a vector in $\mathbb{R}^n$. Then the following are equivalent (if one is true then they are all true, if one is false then they are all false). * The set $\{a_1, \ldots, a_m\}$ are linearly independent. * The vector equation $x_1a_1 + x_2a_2 + \ldots + x_ma_m = b$ has at most one solution. * The linear system $[a_1\ a_2\ \ldots\ a_m|b]$ has at most one solution. * The equation $Ax = b$ has at most 1 solution.

Example: Consider the vectors $a_1 = (1, 7, -2)$, $a_2 = (3, 0, 1)$, and $a_3 = (5, 2, 6)$. Set $A = [a_i]$. Show that the columns of A are linearly independent and that $Ax = b$ has a unique solution for every b in $\mathbb{R}^3$.