The Calculus of Parametric and Polar Equations

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1 Parametric Equations

$$x = f(t)\ ,\ y = g(t)$$

Tangents

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Areas

$$A = \int_a^b F(x)dx$$

Substituting for x and dx to get the integral in terms of t:

$$\frac{dx}{dt} = f'(t)$$

$$dx = f'(t)dt$$

$$A = \int_{\alpha}^{\beta} F(f(t))f'(t)dt$$

$$A = \int_{\alpha}^{\beta} g(t)f'(t)dt$$

Arc Lengths

When $y = F(x)$ arc length is given by

$$L = \int_{\alpha}^{\beta} \sqrt{1 + F'(x)^2}dx$$

For parametric equations this becomes

$$L = \int_{\alpha}^{\beta} \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}dx$$

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

or

$$L = \int_{\alpha}^{\beta} \sqrt{f'(t)^2 + g'(t)^2}\ dt$$

2 Polar Equations

$$r = f(\theta)$$

$$x = r\,cos(\theta)\ ,\ y = r\,sin(\theta)$$

$$x^2 + y^2 = r^2\ ,\ \theta = tan^{-1}\frac{y}{x}$$

Symmetry

1. If the equation is unchanged when θ is replaced by $-\theta$, then the graph is symmetric about the polar axis.

2. If the equation is unchanged when r is replaced by -r, then the graph is symmetric about the pole.

3. If the equation is unchanged when θ is replaced by $(\pi - \theta)$, then the graph is symmetric about the line $\theta = \frac{\pi}{2}$.

Quick Tests for Symmetry and Graphing Aids

1. The graph of $r = f(\sin(\theta))$ is symmetric with respect to the line $\theta = \frac{\pi}{2}$.
2. The graph of $r = f(\cos(\theta))$ is symmetric with respect to the polar axis.
3. Additional aids to graphing: find the θ values where $r = 0$ and $|r|$ is a maximum.

Tangents

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta} sin\theta + rcos\theta}{\frac{dr}{d\theta} cos\theta - rsin\theta}$$

Note, at the pole, where $r = 0$, then $\frac{dy}{dx} = tan\theta$ if $\frac{dr}{d\theta} \neq 0$

Areas

$$A_{sector} = \frac{1}{2} r^2 \theta$$

$$A = \int_a^b \frac{1}{2} [f(\theta)]^2 d\theta = \int_a^b \frac{1}{2} r^2 d\theta$$

The area between curves, where $f(\theta) \geq g(\theta) \geq 0$, is

$$A = \int_a^b \frac{1}{2} [f(\theta)]^2 d\theta - \int_a^b \frac{1}{2} [g(\theta)]^2 d\theta$$

Arc Lengths

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$