ModulePresentationsForCAP

Category R-pres for CAP

2017.09.09

9 September 2017

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Chapter 1

Module Presentations

1.1 Functors

1.1.1 FunctorStandardModuleLeft (for IsHomalgRing)

▷ `FunctorStandardModuleLeft(R)` (attribute)

Returns: a functor

The argument is a homalg ring *R*. The output is a functor which takes a left presentation as input and computes its standard presentation.

1.1.2 FunctorStandardModuleRight (for IsHomalgRing)

▷ `FunctorStandardModuleRight(R)` (attribute)

Returns: a functor

The argument is a homalg ring *R*. The output is a functor which takes a right presentation as input and computes its standard presentation.

1.1.3 FunctorGetRidOfZeroGeneratorsLeft (for IsHomalgRing)

▷ `FunctorGetRidOfZeroGeneratorsLeft(R)` (attribute)

Returns: a functor

The argument is a homalg ring *R*. The output is a functor which takes a left presentation as input and gets rid of the zero generators.

1.1.4 FunctorGetRidOfZeroGeneratorsRight (for IsHomalgRing)

▷ `FunctorGetRidOfZeroGeneratorsRight(R)` (attribute)

Returns: a functor

The argument is a homalg ring *R*. The output is a functor which takes a right presentation as input and gets rid of the zero generators.

1.1.5 FunctorLessGeneratorsLeft (for IsHomalgRing)

▷ `FunctorLessGeneratorsLeft(R)` (attribute)

Returns: a functor

The argument is a homalg ring R. The output is functor which takes a left presentation as input and computes a presentation having less generators.

1.1.6 FunctorLessGeneratorsRight (for IsHomalgRing)

▷ `FunctorLessGeneratorsRight(`*R*`)` (attribute)

Returns: a functor

The argument is a homalg ring R. The output is functor which takes a right presentation as input and computes a presentation having less generators.

1.1.7 FunctorDualLeft (for IsHomalgRing)

▷ `FunctorDualLeft(`*R*`)` (attribute)

Returns: a functor

The argument is a homalg ring R that has an involution function. The output is functor which takes a left presentation *M* as input and computes its Hom(M, R) as a left presentation.

1.1.8 FunctorDualRight (for IsHomalgRing)

▷ `FunctorDualRight(`*R*`)` (attribute)

Returns: a functor

The argument is a homalg ring R that has an involution function. The output is functor which takes a right presentation *M* as input and computes its Hom(M, R) as a right presentation.

1.1.9 FunctorDoubleDualLeft (for IsHomalgRing)

▷ `FunctorDoubleDualLeft(`*R*`)` (attribute)

Returns: a functor

The argument is a homalg ring R that has an involution function. The output is functor which takes a left presentation *M* as input and computes its *Hom(Hom(M, R), R)* as a left presentation.

1.1.10 FunctorDoubleDualRight (for IsHomalgRing)

▷ `FunctorDoubleDualRight(`*R*`)` (attribute)

Returns: a functor

The argument is a homalg ring R that has an involution function. The output is functor which takes a right presentation *M* as input and computes its *Hom(Hom(M, R), R)* as a right presentation.

1.2 GAP Categories

1.2.1 IsLeftOrRightPresentationMorphism (for IsCapCategoryMorphism)

▷ `IsLeftOrRightPresentationMorphism(`*object*`)` (filter)

Returns: `true` or `false`

The GAP category of morphisms in the category of left or right presentations.

1.2.2 IsLeftPresentationMorphism (for IsLeftOrRightPresentationMorphism)

▷ `IsLeftPresentationMorphism(`*`object`*`)` (filter)

Returns: `true` or `false`

The GAP category of morphisms in the category of left presentations.

1.2.3 IsRightPresentationMorphism (for IsLeftOrRightPresentationMorphism)

▷ `IsRightPresentationMorphism(`*`object`*`)` (filter)

Returns: `true` or `false`

The GAP category of morphisms in the category of right presentations.

1.2.4 IsLeftOrRightPresentation (for IsCapCategoryObject)

▷ `IsLeftOrRightPresentation(`*`object`*`)` (filter)

Returns: `true` or `false`

The GAP category of objects in the category of left presentations or right presentations.

1.2.5 IsLeftPresentation (for IsLeftOrRightPresentation)

▷ `IsLeftPresentation(`*`object`*`)` (filter)

Returns: `true` or `false`

The GAP category of objects in the category of left presentations.

1.2.6 IsRightPresentation (for IsLeftOrRightPresentation)

▷ `IsRightPresentation(`*`object`*`)` (filter)

Returns: `true` or `false`

The GAP category of objects in the category of right presentations.

1.3 Constructors

1.3.1 PresentationMorphism (for IsLeftOrRightPresentation, IsHomalgMatrix, IsLeftOrRightPresentation)

▷ `PresentationMorphism(`*`A, M, B`*`)` (operation)

Returns: a morphism in $\mathrm{Hom}(A,B)$

The arguments are an object A, a homalg matrix M, and another object B. A and B shall either both be objects in the category of left presentations or both be objects in the category of right presentations. The output is a morphism $A \to B$ in the the category of left or right presentations whose underlying matrix is given by M.

1.3.2 AsMorphismBetweenFreeLeftPresentations (for IsHomalgMatrix)

▷ `AsMorphismBetweenFreeLeftPresentations(`*`m`*`)` (attribute)

Returns: a morphism in $\mathrm{Hom}(F^r,F^c)$

The argument is a homalg matrix m. The output is a morphism $F^r \to F^c$ in the the category of left presentations whose underlying matrix is given by m, where F^r and F^c are free left presentations of ranks given by the number of rows and columns of m.

1.3.3 AsMorphismBetweenFreeRightPresentations (for IsHomalgMatrix)

▷ `AsMorphismBetweenFreeRightPresentations(m)` (attribute)

Returns: a morphism in $\mathrm{Hom}(F^c, F^r)$

The argument is a homalg matrix m. The output is a morphism $F^c \to F^r$ in the the category of right presentations whose underlying matrix is given by m, where F^r and F^c are free right presentations of ranks given by the number of rows and columns of m.

1.3.4 AsLeftPresentation (for IsHomalgMatrix)

▷ `AsLeftPresentation(M)` (operation)

Returns: an object

The argument is a homalg matrix M over a ring R. The output is an object in the category of left presentations over R. This object has M as its underlying matrix.

1.3.5 AsRightPresentation (for IsHomalgMatrix)

▷ `AsRightPresentation(M)` (operation)

Returns: an object

The argument is a homalg matrix M over a ring R. The output is an object in the category of right presentations over R. This object has M as its underlying matrix.

1.3.6 AsLeftOrRightPresentation

▷ `AsLeftOrRightPresentation(M, l)` (function)

Returns: an object

The arguments are a homalg matrix M and a boolean l. If l is `true`, the output is an object in the category of left presentations. If l is `false`, the output is an object in the category of right presentations. In both cases, the underlying matrix of the result is M.

1.3.7 FreeLeftPresentation (for IsInt, IsHomalgRing)

▷ `FreeLeftPresentation(r, R)` (operation)

Returns: an object

The arguments are a non-negative integer r and a homalg ring R. The output is an object in the category of left presentations over R. It is represented by the $0 \times r$ matrix and thus it is free of rank r.

1.3.8 FreeRightPresentation (for IsInt, IsHomalgRing)

▷ `FreeRightPresentation(r, R)` (operation)

Returns: an object

The arguments are a non-negative integer r and a homalg ring R. The output is an object in the category of right presentations over R. It is represented by the $r \times 0$ matrix and thus it is free of rank r.

1.3.9 UnderlyingMatrix (for IsLeftOrRightPresentation)

▷ `UnderlyingMatrix(A)` (attribute)

Returns: a homalg matrix

The argument is an object A in the category of left or right presentations over a homalg ring R. The output is the underlying matrix which presents A.

1.3.10 UnderlyingHomalgRing (for IsLeftOrRightPresentation)

▷ `UnderlyingHomalgRing(A)` (attribute)

Returns: a homalg ring

The argument is an object A in the category of left or right presentations over a homalg ring R. The output is R.

1.3.11 Annihilator (for IsLeftOrRightPresentation)

▷ `Annihilator(A)` (attribute)

Returns: a morphism in $\mathrm{Hom}(I,F)$

The argument is an object A in the category of left or right presentations. The output is the embedding of the annihilator I of A into the free module F of rank 1. In particular, the annihilator itself is seen as a left or right presentation.

1.3.12 LeftPresentations (for IsHomalgRing)

▷ `LeftPresentations(R)` (attribute)

Returns: a category

The argument is a homalg ring R. The output is the category of free left presentations over R.

1.3.13 RightPresentations (for IsHomalgRing)

▷ `RightPresentations(R)` (attribute)

Returns: a category

The argument is a homalg ring R. The output is the category of free right presentations over R.

1.4 Attributes

1.4.1 UnderlyingHomalgRing (for IsLeftOrRightPresentationMorphism)

▷ `UnderlyingHomalgRing(R)` (attribute)

Returns: a homalg ring

The argument is a morphism α in the category of left or right presentations over a homalg ring R. The output is R.

1.4.2 UnderlyingMatrix (for IsLeftOrRightPresentationMorphism)

▷ `UnderlyingMatrix(alpha)` (attribute)

Returns: a homalg matrix

The argument is a morphism α in the category of left or right presentations. The output is its underlying homalg matrix.

1.5 Non-Categorical Operations

1.5.1 StandardGeneratorMorphism (for IsLeftOrRightPresentation, IsInt)

▷ `StandardGeneratorMorphism(A, i)` (operation)

Returns: a morphism in $\mathrm{Hom}(F,A)$

The argument is an object A in the category of left or right presentations over a homalg ring R with underlying matrix M and an integer i. The output is a morphism $F \to A$ given by the i-th row or column of M, where F is a free left or right presentation of rank 1.

1.5.2 CoverByFreeModule (for IsLeftOrRightPresentation)

▷ `CoverByFreeModule(A)` (attribute)

Returns: a morphism in $\mathrm{Hom}(F,A)$

The argument is an object A in the category of left or right presentations. The output is a morphism from a free module F to A, which maps the standard generators of the free module to the generators of A.

1.6 Natural Transformations

1.6.1 NaturalIsomorphismFromIdentityToStandardModuleLeft (for IsHomalgRing)

▷ `NaturalIsomorphismFromIdentityToStandardModuleLeft(R)` (attribute)

Returns: a natural transformation Id $\to$ StandardModuleLeft

The argument is a homalg ring R. The output is the natural isomorphism from the identity functor to the left standard module functor.

1.6.2 NaturalIsomorphismFromIdentityToStandardModuleRight (for IsHomalgRing)

▷ `NaturalIsomorphismFromIdentityToStandardModuleRight(R)` (attribute)

Returns: a natural transformation Id $\to$ StandardModuleRight

The argument is a homalg ring R. The output is the natural isomorphism from the identity functor to the right standard module functor.

1.6.3 NaturalIsomorphismFromIdentityToGetRidOfZeroGeneratorsLeft (for IsHomalgRing)

▷ `NaturalIsomorphismFromIdentityToGetRidOfZeroGeneratorsLeft(R)` (attribute)

Returns: a natural transformation Id $\to$ GetRidOfZeroGeneratorsLeft

The argument is a homalg ring R. The output is the natural isomorphism from the identity functor to the functor that gets rid of zero generators of left modules.

1.6.4 NaturalIsomorphismFromIdentityToGetRidOfZeroGeneratorsRight (for IsHomalgRing)

▷ `NaturalIsomorphismFromIdentityToGetRidOfZeroGeneratorsRight(R)` (attribute)

Returns: a natural transformation Id $\to$ GetRidOfZeroGeneratorsRight

The argument is a homalg ring R. The output is the natural isomorphism from the identity functor to the functor that gets rid of zero generators of right modules.

1.6.5 NaturalIsomorphismFromIdentityToLessGeneratorsLeft (for IsHomalgRing)

▷ `NaturalIsomorphismFromIdentityToLessGeneratorsLeft(R)` (attribute)

Returns: a natural transformation Id $\rightarrow$ LessGeneratorsLeft

The argument is a homalg ring R. The output is the natural morphism from the identity functor to the left less generators functor.

1.6.6 NaturalIsomorphismFromIdentityToLessGeneratorsRight (for IsHomalgRing)

▷ `NaturalIsomorphismFromIdentityToLessGeneratorsRight(R)` (attribute)

Returns: a natural transformation Id $\rightarrow$ LessGeneratorsRight

The argument is a homalg ring R. The output is the natural morphism from the identity functor to the right less generator functor.

1.6.7 NaturalTransformationFromIdentityToDoubleDualLeft (for IsHomalgRing)

▷ `NaturalTransformationFromIdentityToDoubleDualLeft(R)` (attribute)

Returns: a natural transformation Id $\rightarrow$ FunctorDoubleDualLeft

The argument is a homalg ring R. The output is the natural morphism from the identity functor to the double dual functor in left Presentations category.

1.6.8 NaturalTransformationFromIdentityToDoubleDualRight (for IsHomalgRing)

▷ `NaturalTransformationFromIdentityToDoubleDualRight(R)` (attribute)

Returns: a natural transformation Id $\rightarrow$ FunctorDoubleDualRight

The argument is a homalg ring R. The output is the natural morphism from the identity functor to the double dual functor in right Presentations category.

Chapter 2

Examples and Tests

2.1 Annihilator

Example

```
gap> ZZ := HomalgRingOfIntegersInSingular();;
gap> M1 := AsLeftPresentation( HomalgMatrix( [ [ "2" ] ], ZZ ) );;
gap> M2 := AsLeftPresentation( HomalgMatrix( [ [ "3" ] ], ZZ ) );;
gap> M3 := AsLeftPresentation( HomalgMatrix( [ [ "4" ] ], ZZ ) );;
gap> M := DirectSum( M1, M2, M3 );;
gap> Display( Annihilator( M ) );
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A monomorphism in Category of left presentations of Z
gap> M1 := AsRightPresentation( HomalgMatrix( [ [ "2" ] ], ZZ ) );;
gap> M2 := AsRightPresentation( HomalgMatrix( [ [ "3" ] ], ZZ ) );;
gap> M3 := AsRightPresentation( HomalgMatrix( [ [ "4" ] ], ZZ ) );;
gap> M := DirectSum( M1, M2, M3 );;
gap> Display( Annihilator( M ) );
12

A monomorphism in Category of right presentations of Z
```

2.2 Intersection of Submodules

Example

```
gap> Q := HomalgFieldOfRationalsInSingular();;
gap> R := Q * "x,y";
Q[x,y]
gap> F := AsLeftPresentation( HomalgMatrix( [ [ 0 ] ], R ) );
<An object in Category of left presentations of Q[x,y]>
gap> I1 := AsLeftPresentation( HomalgMatrix( [ [ "x" ] ], R ) );;
gap> I2 := AsLeftPresentation( HomalgMatrix( [ [ "y" ] ], R ) );;
gap> Display( I1 );
x

An object in Category of left presentations of Q[x,y]
gap> Display( I2 );
y

```

```
An object in Category of left presentations of Q[x,y]
gap> eps1 := PresentationMorphism( F, HomalgMatrix( [ [ 1 ] ], R ), I1 );
<A morphism in Category of left presentations of Q[x,y]>
gap> eps2 := PresentationMorphism( F, HomalgMatrix( [ [ 1 ] ], R ), I2 );
<A morphism in Category of left presentations of Q[x,y]>
gap> kernelemb1 := KernelEmbedding( eps1 );
<A monomorphism in Category of left presentations of Q[x,y]>
gap> kernelemb2 := KernelEmbedding( eps2 );
<A monomorphism in Category of left presentations of Q[x,y]>
gap> P := FiberProduct( kernelemb1, kernelemb2 );;
gap> Display( P );
(an empty 0 x 1 matrix)

An object in Category of left presentations of Q[x,y]
gap> pi1 := ProjectionInFactorOfFiberProduct( [ kernelemb1, kernelemb2 ], 1 );
<A monomorphism in Category of left presentations of Q[x,y]>
gap> composite := PreCompose( pi1, kernelemb1 );
<A monomorphism in Category of left presentations of Q[x,y]>
gap> Display( composite );
x*y

A monomorphism in Category of left presentations of Q[x,y]
```

2.3 Koszul Complex

Example

```
gap> Q := HomalgFieldOfRationalsInSingular();;
gap> R := Q * "x,y,z";;
gap> M := HomalgMatrix( [ [ "x" ], [ "y" ], [ "z" ] ], 3, 1, R );;
gap> Ml := AsLeftPresentation( M );;
gap> eps := CoverByFreeModule( Ml );;
gap> iota1 := KernelEmbedding( eps );;
gap> Display( iota1 );
x,
y,
z

A monomorphism in Category of left presentations of Q[x,y,z]
gap> Display( Source( iota1 ) );
0, -z,y,
-y,x, 0,
-z,0, x

An object in Category of left presentations of Q[x,y,z]
gap> pi1 := CoverByFreeModule( Source( iota1 ) );;
gap> d1 := PreCompose( pi1, iota1 );;
gap> Display( d1 );
x,
y,
z

A morphism in Category of left presentations of Q[x,y,z]
```

```
gap> iota2 := KernelEmbedding( d1 );;
gap> Display( iota2 );
0, -z,y,
-y,x, 0,
-z,0, x

A monomorphism in Category of left presentations of Q[x,y,z]
gap> Display( Source( iota2 ) );;
x,z,-y

An object in Category of left presentations of Q[x,y,z]
gap> pi2 := CoverByFreeModule( Source( iota2 ) );;
gap> d2 := PreCompose( pi2, iota2 );;
gap> Display( d2 );
0, -z,y,
-y,x, 0,
-z,0, x

A morphism in Category of left presentations of Q[x,y,z]
gap> iota3 := KernelEmbedding( d2 );;
gap> Display( iota3 );
x,z,-y

A monomorphism in Category of left presentations of Q[x,y,z]
gap> Display( Source( iota3 ) );
(an empty 0 x 1 matrix)

An object in Category of left presentations of Q[x,y,z]
gap> pi3 := CoverByFreeModule( Source( iota3 ) );;
gap> d3 := PreCompose( pi3, iota3 );;
gap> Display( d3 );
x,z,-y

A morphism in Category of left presentations of Q[x,y,z]
gap> N := HomalgMatrix( [ [ "x" ] ], 1, 1, R );;
gap> Nl := AsLeftPresentation( N );;
gap> d2Nl := TensorProductOnMorphisms( d2, IdentityMorphism( Nl ) );;
gap> d1Nl := TensorProductOnMorphisms( d1, IdentityMorphism( Nl ) );;
gap> IsZero( PreCompose( d2Nl, d1Nl ) );
true
gap> cycles := KernelEmbedding( d1Nl );;
gap> boundaries := ImageEmbedding( d2Nl );;
gap> boundaries_in_cyles := LiftAlongMonomorphism( cycles, boundaries );;
gap> homology := CokernelObject( boundaries_in_cyles );;
gap> LessGenFunctor := FunctorLessGeneratorsLeft( R );;
gap> homology := ApplyFunctor( LessGenFunctor, homology );;
gap> StdBasisFunctor := FunctorStandardModuleLeft( R );;
gap> homology := ApplyFunctor( StdBasisFunctor, homology );;
gap> Display( homology );
z,
y,
x
```

```
An object in Category of left presentations of Q[x,y,z]
```

2.4 Closed Monoidal Structure

Example
```
gap> R := HomalgRingOfIntegers( );;
gap> M := AsLeftPresentation( HomalgMatrix( [ [ 2 ] ], 1, 1, R ) );
<An object in Category of left presentations of Z>
gap> N := AsLeftPresentation( HomalgMatrix( [ [ 3 ] ], 1, 1, R ) );
<An object in Category of left presentations of Z>
gap> T := TensorProductOnObjects( M, N );
<An object in Category of left presentations of Z>
gap> Display( T );
[ [  3 ],
  [  2 ] ]

An object in Category of left presentations of Z
gap> IsZero( T );
true
gap> H := InternalHomOnObjects( DirectSum( M, M ), DirectSum( M, N ) );
<An object in Category of left presentations of Z>
gap> Display( H );
[ [  -4,  -2 ],
  [   2,   2 ] ]

An object in Category of left presentations of Z
gap> alpha := StandardGeneratorMorphism( H, 1 );
<A morphism in Category of left presentations of Z>
gap> l := LambdaElimination( DirectSum( M, M ), DirectSum( M, N ), alpha );
<A morphism in Category of left presentations of Z>
gap> IsZero( l );
false
gap> Display( l );
[ [  0,   0 ],
  [  1,  0 ] ]

A morphism in Category of left presentations of Z
gap> alpha2 := StandardGeneratorMorphism( H, 2 );
<A morphism in Category of left presentations of Z>
gap> l2 := LambdaElimination( DirectSum( M, M ), DirectSum( M, N ), alpha2 );
<A morphism in Category of left presentations of Z>
gap> IsZero( l2 );
false
gap> Display( l2 );
[ [  1,   0 ],
  [  0,  0 ] ]

A morphism in Category of left presentations of Z
```

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