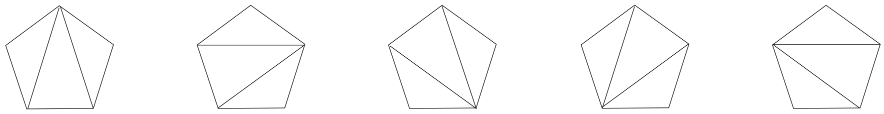
Catalania starters

1. How many ways are there of cutting a regular heptagon with four non-crossing lines into triangles? For example, there are five ways for a pentagon:



2. How many ways can the "word" $a_1 + a_2 + \cdots + a_6$ be (legally) parenthesised with 5 pairs of parentheses, so that there are precisely three things inside each (balanced) pair, each being either one of the a_i, a +-sign or a parenthesised expression? An example will make this clearer. For $a_1 + a_2 + a_3 + a_4$ there are five ways:

$$(((a_1 + a_2) + a_3) + a_4) \qquad ((a_1 + a_2) + (a_3 + a_4)) \qquad ((a_1 + (a_2 + a_3)) + a_4)$$
$$(a_1 + ((a_2 + a_3) + a_4)) \qquad (a_1 + (a_2 + (a_3 + a_4)))$$

3. Alice and Bert are the two candidates in the Catalan local elections. In a closely fought contest, the result was—a draw! Interestingly though, the whole way through the count, Alice never trailed Bert. If there were 10 votes in all, in how many ways could they have been counted? (We do not distinguish between the five Alice votes or between the five Bert votes.) For example, if there were only 6 votes, they could have been counted in five ways:

AAABBB AABABB AABBAB ABAABB ABABAB

4. Consider a sequence of positive integers $a_1, a_2, \ldots, a_5$ satisfying the following two conditions:
   (i) $a_{i-1} \leqslant a_i$ for $i = 2, 3, \ldots, 5$, and
   (ii) $a_i \leqslant i$ for $i = 1, 2, \ldots, 5$.

   How many such sequences are there? As an example, if '5' were replaced by '3' throughout the question, the answer would be five, the sequences being:

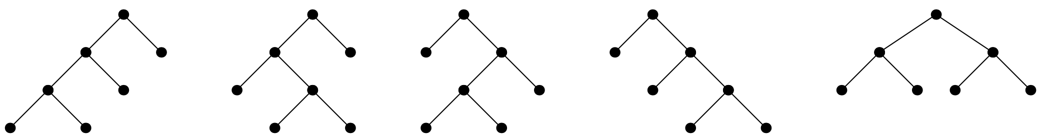
$$1,1,1 \quad 1,1,2 \quad 1,1,3 \quad 1,2,2 \quad 1,2,3$$

A few extras

5. Why are the answers to questions 1–4 all the same?

6. What would the answer be to question 3 if Alice had won by one vote (and there were 11 cast in all)? And what if by two votes, with 12 cast? Can you generalise this result?

7. A *binary tree* consists of either a point or a point with two binary trees emanating from it (as in the picture below). How many binary trees are there with 11 points? With 7 points, there are five:



And there's more on the other side!

8. (This is quite a hard one; it is based on IMO 1996/6, but with sufficient hints to make it approachable.)

   Let p, q, n be three positive integers with $p+q<n$. Let $(x_0, x_1, \ldots, x_n)$ be an $(n+1)$-tuple of integers satisfying the following conditions:
   (i) $x_0 = x_n = 0$.
   (ii) For each i with $1 \leqslant i \leqslant n$, either $x_i - x_{i+1} = p$ or $x_i - x_{i+1} = -q$.

   Try answering the following questions:
   (a) How many such sequences are there? *(Hint: what would be the case if p and q were coprime?)*
   (b) What possible values can $x_{(p+q)+i} - x_i$ take, where $0 \leqslant i \leqslant n-(p+q)$?
   (c) Show that there exist indices $i<j$ with $(i,j) \neq (0,n)$, such that $x_i = x_j$.
   (d) Challenge (and I don't know the answer myself): What would be the answer to (a) if we were also told that $x_i \geqslant 0$ for each i? (Feel free to solve the simpler question of the case where $n = p+q$ and p and q are coprime! I don't know this, either.)

9. Several cities in the USA have a very simple street pattern: their roads go north-south ("Avenues") and east-west ("Streets"). Imagine that you are standing at the corner of 1st St. and 1st Ave. You are trying to get to the corner of 6th St. and 6th Ave. Obviously, you are only going to walk north and east (why go a long way round?). How many different ways are there of reaching your destination? What if you wanted to reach the corner of mth St. and nth Ave? What would the answers be if you weren't allowed to cross the diagonal line shown, where we assume $n \geqslant m$ in the latter case? (The roads stop there because the city is built on a lake.)