COMPLEX VARIABLES - PROBLEM SET 1

MATHCAMP '99

1. Calculate the following quantities:
   (a) i^i
   (b) $\log(-5)$
   (c) Find all the cube roots of $1+i$
   (d) $\frac{3+4i}{2-2i}$
   (e) $(1-i\sqrt{3})^{18}$
   (f) $\sin -i$
   (g) $\tanh(4+4i)$
   (h) Find all the 4^{th} roots of -1
2. One needs to use calculus to actually prove $e^{i\theta} = \cos\theta + i\sin\theta$. However we can look at various properties of real exponentials and show they still work with this imaginary exponential.
   (a) One property we know of real exponentials is that $e^r e^s = e^{r+s}$. Show this works when r and s are imaginary numbers.
   (b) Another property is that $e^{-s} = \frac{1}{e^s}$. Show this works for the imaginary exponential.
   (c) If one applies the rule $(e^r)^n = e^{rn}$ to the imaginary exponential, one gets the De Moivre's theorem $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, where n is a natural number. Prove this relation.
3. Let's look at where entire regions of the complex plane go under certain functions.
   (a) Take the rectangle that goes from $(0,0)$ to $(2\pi, 2\pi)$ (another way of writing this is all complex numbers such that $0 \leq \Re z \leq 2\pi$ and $0 \leq \Im z \leq 2\pi$.) Apply e^z to this figure. What does the set look like now?
   (b) Look at the line $\Im z = 1$ (aka the line $y = 1$). Where does this line go when you apply $\sin z$ to it? What about the line $\Re z = 1$ $(x = 1)$? What about the line $x = 0$?
   (c) What is the set $|z - i| \leq 1$? What happens to it when you apply $2/z$ to it?
   (d) Can you think of some mappings that send the unit circle centered at the origin back onto itself?
   (e) Can you think of a mapping that maps the upper half plane ($\Im z > 0$) onto a disk?

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