COMPLEX VARIABLES – GENERAL INFO

MATHCAMP '99

$i^2 = -1$. All else follows.
z is always our complex variable in the following sections. And maybe w.

1. Representation

1.1. Standard Form.

- The standard form for a complex variable $z = a + bi$, where $a, b \in \mathbb{R}$. If we plot this number on the complex plane $\mathbb{C}$, it's just the same as plotting (a, b) in $\mathbb{R}^2$.
- The x-axis is called the real axis, the y-axis is called the imaginary axis.
- The x-coordinate is then the *real part* of z and is denoted as $\Re z = a$
- The y-coordinate is the *imaginary part* of z and is denoted as $\Im z = b$ The *complex conjugate* of z is $\overline{z} = a - bi$, which one can get by reflecting z over the real axis.
- Cute trick: to get the complex conjugate of a complicated complex expression, turn all the i's to $-i$s and all the variables into their complex conjugates.

1.2. Polar Form.

- The basis of the polar form is Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$. So if we have a complex variable z, its polar form is $re^{i\theta} = r\cos\theta + ir\sin\theta$, which is the standard form.
- r is the *modulus* of z, denoted as $r = |z| = \sqrt{z\overline{z}} = \sqrt{a^2 + b^2}$. This is simply the Euclidean distance between z and the origin. In the reals, the modulus is the same as absolute value.
- θ is the *argument* of z, denoted as $\arg z$. $\tan\theta = \frac{\Im z}{\Re z}$, and be careful as to its quadrant. Obviously, one can add a multiple of 2π to θ and get the same result. So there's something called the *principal argument* of z, denoted as $\operatorname{Arg} z$ which is the θ that works and $0 \leq \theta < 2\pi$. This is important when working with certain inverse functions.

2. Functions

2.1. Arithmetic.

- Addition and subtraction are as one thinks, just add(subtract) the real parts together, and add(subtract) the imaginary parts. Standard form is best for this.
- Multiplication and division can be done in either standard or polar form. In standard form, one just expands out a multiplication and remembers to change i^2 to -1. If one is dividing by a complex number, one should multiply the numerator and denominator by its complex conjugate, so that you're dividing by a real number.
- However, multiplication and division is much easier in polar form: if $z = re^{i\theta}, w = se^{i\phi}$, then $wz = rse^{i(\theta+\phi)}$ and $\frac{w}{z} = \frac{s}{r}e^{i(\phi-\theta)}$. In multiplication, one rotates counterclockwise by the argument and multiplies the modulus. Division looks pretty much the same in polar form.

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2.2. Powers and Exponentials.

- When taking powers of a complex number, it's best to use the polar form, because one an take advantage of the fact that $e^{i\theta}$ follows the same rules as real exponentials. So $z^n = (re^{i\theta})^n = r^n e^{ni\theta}$ and $\frac{1}{z} = z^{-1} = \frac{1}{r}e^{-i\theta}$.
- What if we take $z^{\frac{1}{n}}$? One can get more than one answer, because θ is not unique. One gets the principal root if you start with θ as the principal argument of z.
- One uses the Euler formula in dealing with exponential functions. $e^z = e^{a+bi} = e^a e^{ib}$. So one starts out with z in standard form, and end up with a complex number in polar form. One does switch between the forms rather often, depending on how one calculates the functions one wants.
- So one expects that to take the log of a complex number, one starts in polar form and ends in standard form (one would be right in that expectation.) $\log(z) = \log(re^{i\theta}) = \log r + i\theta$. Again, since θ is not unique, one can get different values for log. This is another case where one might be interested in the principal value.
- Finally, if one wants to take a complex number to an arbitrary complex power, do this: $z^w = (e^{\log z})^w = e^{w \log z}$. Again, this answer is not necessarily unique.

2.3. Trigonometric.

- The Euler formula works for all complex numbers θ, not just real angles. Therefore, one can solve for the complex trigonometric functions.
- $\cos z = \frac{e^{iz}+e^{-iz}}{2} = \cosh(iz)$
- $\sin z = \frac{e^{iz}-e^{-iz}}{2i} = -i\sinh(iz)$
- The other trigonometric functions, of course, are defined in terms of sine and cosine. In these cases, it's usually best to have z in standard form.

3. Things to Think About

3.1. Problems with inverses and rules. So in class, I showed that $-1 = i^2 = \sqrt{-1}\sqrt{-1} = \sqrt{-1*-1} = \sqrt{1} = 1$ using the usual rules for combining square roots. However, the rule for combining square roots does work some of the time in complex numbers. When can you use it? When does it go wrong? Obviously something about $z^{\frac{1}{2}}$ not being unique is affecting this. Consider another function that gives nonunique answers - the log. How are rules for combining logs affected?

3.2. Locus of points. Because we have a standard, Cartesian form for complex numbers, and a polar form, perhaps we can define certain sets in the plane in an elegant way. For example, a circle C can be defined as: $C = \{z : |z - z_0| = r\}$, where z_0 is a given complex number (the center of the circle) and r is the length of the radius. How does one define lines? Ellipses? Other conic sections? Spirals? What other shapes can you think of?

3.3. Functions as maps. So we've seen that with multiplication by a certain complex number, one gets a multiplication of distance from origin and a rotation. So if one starts with a circle centered at the origin, one gets a different sized circle (you don't notice the rotation.) What happens to a circle not centered at the origin? What happens to lines under multiplication by a complex constant?
The questions I just asked involved taking some set, applying a function to every point in the set, and seeing what the end result looked like. Can one think of functions that take circles to lines? Lines to circles? Rectangular areas to disks? Can one send a point out to infinity (and what does that mean?) What happens to areas of sets under these mappings? Do they get turned inside out? What about angles intersecting lines, or a line intersecting a circle? Do angles between lines get larger or smaller under these mappings?