Problems on Generating Functions

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These problems are intended both for practising the ideas we have looked at in class and for introducing some new ones. We hope you find them enlightening!

1. (a) Find the sequence (a_n) such that

$$\frac{x}{1-5x+6x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

(Hint: First rewrite the fraction in a more manageable form.)

(b) Which sequences have generating functions of the form

$$\frac{Cx+D}{1-5x+6x^2}$$

(where C and D are constants)?

(c)* Which sequences have generating functions of the form

$$\frac{P(x)}{1-5x+6x^2}$$

where $P(x)$ is a polynomial?

2. The method used for simplifying the fraction in problem 1 will not work for

$$\frac{1}{1-2x+x^2} = \frac{1}{(1-x)^2}.$$

Nevertheless, it is possible to find a formal power series $G(x) = \sum_{n\geqslant 0} b_n$ with $G(x) = 1/(1-2x+x^2)$, that is with

$$(1-2x+x^2)(b_0 + b_1x + b_2x^2 + \cdots) = 1, \qquad (*)$$

as follows:

(a) Compute the values of b_0, b_1, b_2 and b_3 which will make $(*)$ correct.

(b) Guess what b_n should be in general, and show that $(*)$ holds for this choice of the b_n.

3. In this question, we extend the technique used in problem 2.

(a) Suppose that we are given *any* formal power series $F(z) = \sum_{n\geqslant 0} f_n z^n$ with $f_0 = 1$. Show that there is a unique formal power series $G(z) = \sum_{n\geqslant 0} g_n z^n$ with $F(z)G(z) = 1$. (How does this relate to problem 2?) Hint: don't try to look for a general formula, but simply show that you can find the g_n recursively, first g_0, then g_1 using the newly found g_0, and in general you can find g_n once you have found $g_0, g_1, \ldots, g_{n-1}$.

(b) If all of the f_n are integers, what can you deduce about the g_n?

(c) What happens if $f_0 \neq 1$?

4. In this question, we will practise some manipulations of formal power series. We let $F(z) = \sum_{n\geqslant 0} f_n z^n$ and $G(z) = \sum_{n\geqslant 0} g_n z^n$ throughout the question.

(a) Let c be a constant. Show that

$$F(cz) = \sum_{n\geqslant 0} c^n f_n z^n.$$

Deduce that

$$\frac{1}{1-cz} = \sum_{n\geqslant 0} c^n z^n.$$

(This approach saves us from having to deduce the formal power series for $1/(1-cz)$ directly.)

(b) A reminder of the definition of derivative: we define

$$DF(z) = \sum_{n\geqslant 1} n f_n z^{n-1} = \sum_{n\geqslant 0} (n+1) f_{n+1} z^n.$$

We often write DF in place of $DF(z)$ if the variable is not changing; this reduces clutter and makes things easier to read. As this is a definition, we do not yet know anything about its properties. So show, using the definitions only, that (i) $D(cF) = c(DF)$ (where c is a constant), (ii) $D(F+G) = DF+DG$, (iii) $D(F(cz)) = c(DF)(cz)$ (this means: if you substitute cz for z, then differentiate, it's the same as differentiating, then substituting cz and finally multiplying by c), and (iv) $D(FG) = (DF)G + F(DG)$. (Properties (i) and (ii) show that D is linear, (iv) is the essential property of differentiation and (iii) is a special case of the chain rule.)

(c) Let $F(z) = 1/(1-z)$. Find $DF(z)$. (You may wish to use the result of problem 2. Alternatively, you may wish to use the result of (b)(iv) above.)

5. (a) In problem 2, we found the power series expansion for $1/(1-z)^2$. Can you find the power series expansion for $1/(1-z)^3$ and $1/(1-z)^4$? You will probably find it useful to consider the identity

$$\frac{1}{(1-z)^r} = \frac{1}{(1-z)}\frac{1}{(1-z)^{r-1}}.$$

(b) Can you spot a pattern in (a) and guess the result for $1/(1-z)^r$ for r a positive integer? Can you prove this? And can you give a combinatorial explanation of your answer?

6. (a) Let $G(z) = (1-z)^n$. Find $DG(z)$. (Remember that you don't know a chain rule yet, and we haven't even talked about whether you can have a function of a function!)

(b) Let $F(z) = 1/(1-z)^r$. Find $DF(z)$ when $r = 2$ and when $r = 3$. (Compare problem 4(c).)

(c) Find the derivatives of $z^s/(1-z)^r$ for $r = 1$, 2 and 3.

7. (a) Suppose that $F(x) = \sum_{n=0}^{\infty} a_n x^n$, with $a_0 = 1$, is a formal power series such that

$$(F(x))^2 = 1 + x.$$

Find a_1, a_2, a_3 and a_4.

(b) Find a recursive expression for a_n, giving a_n in terms of the preceding a_i's. (You might want to distinguish between the cases n odd and n even.)

(c)* Show that $2^{2n-1}a_n$ is an integer for all n.

(d)* Can you find a connection between the coefficients a_n and the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$?

8.* (This question requires a little experience with limits.) In this question, we try to give a justification for the results of problem 2.

(a) Let $t \neq 1$ be any constant. Using the method of problem 1, find a formula for a_n in terms of t, where

$$\frac{1}{(1-x)(1-tx)} = \sum_{n\geqslant 0} a_n x^n.$$

(b) By taking the limit as $t \to 1$, assuming that the limit of the power series can be taken coefficient by coefficient, give another 'proof' of the result of problem 2.