Measure and Category

Here are some of the basic definitions we will be using. All the propositions which are not definitions (such as remarks, theorems, etc.) are left to be proven by the reader. They are really easy, and it is essential to work on them to build a correct feeling of the definitions.

Definitions.

- A subset S of $\mathbb{R}$ is said to be *open* if for any $x \in S$, there is an open interval (a,b) contained in S such that $a < x < b$.
- A point x is said to be a *limit point* of S if for any open interval I containing x, we have $I \cap S \neq \emptyset$.
- A set S is said to be *closed* if all limit points of S are in S.
- *Interior* of a set S is defined to be the union of all open sets contained in S. We denote it by $\dot{S}$.
- *Closure* of a set S is defined to be the intersection of all closed sets containing S. We denote it by $\overline{S}$.

Remarks.

- Arbitrary intersections of closed sets are closed. (not true when *closed* replaced by *open*)
- Arbitrary unions of open sets are open. (not true when *open* replaced by *closed*)
- $\overline{S}$ is the smallest closed set containing S.
- $\dot{S}$ is the biggest open set contained in S.

Theorem (Cantor). For any sequence $\{a_n\}$ of real numbers and for any interval I there exists a point p in I such that $p \neq a_n$ for every n.

This implies that no interval is countable.

Definitions.

- Let S and T be two sets. S is said to be dense in T if for any $x \in T$ and for any open interval I containing x, we have $I \cap S \neq \emptyset$. S is said to be dense, if it is dense in $\mathbb{R}$.
- A subset S of $\mathbb{R}$ is said to be nowhere dense if it is dense in no interval.

Equivalently, every interval has a subinterval not intersecting S.

Equivalently, the complement S^c of S contains a dense open set.

Equivalently, $\overline{S}$ has empty interior.

Needless to say, you are supposed to prove the equivalence of these four sentences.

Theorem. Any subset of a nowhere dense set is nowhere dense. The union of finite number of nowhere dense sets is nowhere dense. The closure of a nowhere dense set is nowhere dense.

Definitions. A set is said to be of *first category* if it can be represented as a countable union of nowhere dense sets. A subset of $\mathbb{R}$ that can not be so represented is said to be of second category.

Theorem (Baire). The complement of any set of first category on the line is dense. No interval in $\mathbb{R}$ is of first category. The intersection of any sequence of dense open sets is dense.

Note that Baire's theorem implies Cantor's.

Thorem. Any subset of a set of first category is of first category. The union of any countable family of first category sets is of first category.

Definition. The length of any interval I is denoted by $|I|$. A set $S \subset \mathbb{R}$ is called a *nullset* if for each $\epsilon > 0$ there exists a sequence of intervals I_n such that $A \subset \bigcup I_n$ and $\sum |I_n| < \epsilon$.

Remarks.

- A one point set is a nullset.
- A countable set is a nullset.
- A countable union of nullsets is a nullset.

Theorem (Borel). If a finite or infinite sequence of intervals I_n covers an interval I, then $\sum |I_n| \geq |I|$.

Note that this implies that no interval is a nullset; it therefore provides another proof of Cantor's theorem.

Definition. Let $C_0 = [0, 1]$. Define C_1 to be $C_0 - (1/3, 2/3)$. For $n \geq 2$, define C_n to be the set obtained by removing middle thirds of all intervals in C_{n-1}. Let $C = \bigcap C_n$. C is called the *Cantor set.*

Prove that

- C is uncountable.
- C is a nullset.
- C is of first category (in fact nowhere dense).

Theorem. $\mathbb{R}$ can be decomposed into two complementary sets A and B such that A is of first category and B is a nullset.

Corollary. Every subset of the line can be represented as the union of a nullset and a set first category.