SOME PROBLEMS IN PROJECTIVE GEOMETRY

1. **The Dual of Pappus' Theorem**
   Recall that Pappus' Theorem says that if l and m are two distinct lines, and A, B, and C are distinct points on l and D, E, and F are distinct points on m such that none of these points lies on $l \cap m$, then the three points
   $P := AE \cup BD$
   $Q := AF \cup CD$
   $R := BF \cup CE$
   are collinear.
   State the dual of Pappus' Theorem, and draw a diagram of it.
2. Prove that if a collineation $\mathcal{C}$ fixes three points on a given line l, then $\mathcal{C}$ fixes every point on l.
3. **Perspectivities**
   Let $\mathcal{C}$ be a collineation with the property that there exists a line l such that for every point $P \in l$, $\mathcal{C}(P) = P(*)$.
   (a) What is the dual of the statement $(*)$?
   (b) Prove that the statement implies its dual, and vice versa.
   These types of collineations are called *perspectivities*. When the point V from the dual statement lies on l, then we call $\mathcal{C}$ an *elation*. When V does not lie on l, we call $\mathcal{C}$ a *homology*.
   We call V the *centre* of $\mathcal{C}$, and we call l the *axis* of $\mathcal{C}$ (sound familiar?).
4. Prove that if a perspectivity $\mathcal{C}$ fixes a point $P \neq V, P \notin l$ (that is: $\mathcal{C}(P) = P$, then $\mathcal{C}$ fixes every point in the plane.
5. Let $\mathcal{H}$ be a homology with centre V, axis l, and with the property $\mathcal{H}(P) = P'$ for some P, P' in the plane. Suppose further that none of V, l, P and P' lie at infinity. Show how, for every point Q in the Euclidean plane, you can use the above information to construct $\mathcal{H}(Q)$.
6. **Luu's Theorem**
   Pick 4 arbitrary points A,B,C and D, no three of which are collinear. Let X,Y,Z be the intersection points of AB and CD, AD and BC, and AC and BD. Let P and Q be the intersection points of XY and BD, and YZ and CD. Then XZ, PQ, and BC are concurrent.
   A few words about Luu's Theorem: Luu's Theorem was, in fact, neither proposed nor proven by Viêt-Trung Luu, for whom it is named; rather, it was both discovered and proven by Yazdani, a contemporary of his. Luu's Theorem is not alone in this regard in the history of mathematics; he is preceded, for instance, by l'Hôpital, whose theorem was actually discovered by one of the Bernouilli brothers.
   Luu and Yazdani were contemporaries at the same university, though the latter was considerably more prolific than the lazier Luu. Time and time again Luu had requested of Yazdani that he name a theorem after him, but

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Yazdani, not surprisingly, consistently refused. Luu was particularly fond of the theorem cited above, and badgered Yazdani consistently for it.

As had previously been the case, Yazdani was not to permit his friend to steal this theorem. One day, thugh, Yazdani let his guard down, and became somewhat intoxicated. Luu cleverly approached his friend at this time, and the inebriated Yazdani responded, ”Look, you ain't getting *nothing* from me, man...and what's that you said about my sister? Wanna take this outside?”

Luu seized the opportunity, and declared, ”I challenge you to a duel! Let us fight to the death!”

Yazdani, alas, was fit, and able, but too drunk to take advantage of either. The fight lasted all of thirteen seconds, and Yazdani fell to his death. The theorem now bears Luu's name alone.

Yazdani is remembered for his theorem, but above all, for his inopportune drunkenness.