Expressing Functions as Power Series

Kamlapati Khalsa

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The Beginning

For a geometric series, $a_n = a_1 r^{n-1}$:

$$S = \sum_{n=1}^{k} a_n = a_1 \sum_{n=1}^{k} r^{n-1} = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{k-1}$$

$$S = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{k-1}$$

$$rS = a_1 r + a_1 r^2 + a_1 r^3 + a_1 r^4 \cdots + a_1 r^{k-1} + a_1 r^k$$

Subtracting the last equation form the prior one, we get

$$S - rS = a_1 - a_1 r^k$$

$$S = \frac{a_1(1 - r^k)}{1 - r}$$

Taking the limit of S as k approaches infinity

$$\lim_{k \to \infty} S = \frac{a_1}{1 - r}, \text{ for } |r| < 1$$

So for $a_1 = 1$, $|x| < 1$

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

This can also be found via polynomial long division. It is worth noting that the domain of the function $1/(1 - x)$ is all real numbers, $x \in \Re$, while the domain of the infinite series is $-1 < x < 1$.

Different values can be substituted for x to find other series. (Many of these can also be found by polynomial long division.)

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n = 1 - x + x^2 - x^3 + x^4 + \dots$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \dots$$

$$\frac{1}{2+x} = \frac{1}{2(1+\frac{x}{2})} = \frac{1}{2(1-(-\frac{x}{2}))} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n$$

The last series converges for $|-\frac{x}{2}| < 1$, or $|x| < 2$.

$$\frac{1}{x} = \frac{1}{1-(-x+1)} = \frac{1}{1-[-(x-1)]} = \sum_{n=0}^{\infty}[-(x-1)]^n = \sum_{n=0}^{\infty}(-1)^n(x-1)^n$$

In general a power series centered on a has the form

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

$$= c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

We can ask, is it possible to rewrite a function so as to move the center, and hence the domain?

For the function $f(x) = 1/(1-x)$ centered on another interval, let's say not $x = 0$ or $-1 < x < 1$, but rather at 3, we can rearrange the f(x) to contain the term $(x-3)$.

$$\frac{1}{1-x} = \frac{1}{(-2)-(x-3)} = \frac{-1/2}{1-[(-\frac{1}{2})(x-3))]}$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}[(-\frac{1}{2})(x-3)]^n = -\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{(x-3)^n}{2^n}$$

This has a radius of convergence of $\mathbf{R} = 2$ and an interval of convergence $x \in (1,5)$.

Derivatives and Integrals

We can use derivatives and integrals to find other series. Take the derivative or the integral of the left hand side of any of the above equations, and use the power rule on the right hand side.

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{n=0}^{\infty}(n+1)x^n$$

$$\frac{d}{dx}\left(\frac{1}{(1-x)^2}\right) = \frac{2}{(1-x)^3} = 2+6x+12x^2+20x^3+\cdots = \sum_{n=0}^{\infty}(n+1)(n+2)x^n$$

$$\frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + \cdots = \sum_{n=0}^{\infty}\binom{n+2}{2}x^n$$

$$\int \frac{1}{1-x}dx = -\ln(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots = \sum_{n=1}^{\infty}\frac{x^n}{n}$$

$$\int \frac{1}{1+x}dx = \ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

Definite integrals can be used to evaluate certain infinite series. For instance, taking the last integral we get

$$\int_0^1 \frac{1}{1+x}dx = \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots\right)\Big|_0^1$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln(2)$$

Similarly

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$$

$$\int \frac{1}{1+x^2}\,dx = \arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \arctan(x)\Big|_0^1 = \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right)\Bigg|_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

Power series provide one of the first clues that any function can be written as an infinite series, and can be approximated by a polynomial. This knowledge is historically important since it was one of the first ways to calculate functions such as the log or cosine. In fact it is still important today as calculators and computers use this approach.

Binomial Series

$$(x+y)^n = x^n + nx^{n-1}y + \cdots + {}_nC_r x^{n-r} y^r + \cdots + nxy^{n-1} + y^{n-1}$$

where

$${}_nC_r = \frac{n!}{(n-r)!\,r!}$$

for n=positive integers.

The Maclaurin series for $f(x) = (1+x)^k$ is

$$\begin{aligned}
&f(x) = (1+x)^k && f(0) = 1 \\
&f'(x) = k(1+x)^{k-1} && f'(0) = k \\
&f''(x) = k(k-1)(1+x)^{k-2} && f''(0) = k(k-1) \\
&f^{(n)}(x) = k(k-1)(k-2)\cdots(k-n+1)(1+x)^{k-n} && f^{(n)}(0) = k(k-1)\cdots(k-n+1)
\end{aligned}$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

For any real number k and $|x| < 1$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

More Binomial Series

The binomial coefficient can be written two ways.

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

The right hand expression works whether n is a positive integer or not, so for general real numbers we'll use that as the definition.

For example, here there are four factors in the numerator of the following expression because we are calculating the coefficient of the 5^{th} term, counting from zero.

$$\binom{1/2}{4} = \frac{(1/2)(-1/2)(-3/2)(-5/2)}{4!} = -\frac{5}{128}$$

The number of factors in the numerator will always equal k when n is not an integer. When k is zero, the numerator is one.

For the simple case with the term $(1+x)$, we get

$$(1+x)^r = \sum_{i=0}^{\infty} \binom{r}{i} (1)^{r-i} x^i$$

Since $(1)^{r-i} = 1$, this simplifies to what is known as Newton's Binomial Theorem.

$$(1+x)^r = \sum_{i=0}^{\infty} \binom{r}{i} x^i$$

for any real r, and $|x| < 1$.

If r is a positive integer and i>r then $\binom{r}{i}$ contains a factor $(r-k)$, the term will equal zero for r>i, reducing to the ordinary Binomial Theorem.

Some examples.

$$(1+x)^{-1} = \binom{-1}{0}x^0 + \binom{-1}{1}x^1 + \binom{-1}{2}x^2 + \binom{-1}{3}x^3 + \ldots$$

$$= 1 + \frac{(-1)}{1!}x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \ldots$$

$$= 1 - x + x^2 - x^3 + \ldots$$

$$(1+x)^{1/2} = \binom{1/2}{0}x^0 + \binom{1/2}{1}x^1 + \binom{1/2}{2}x^2 + \binom{1/2}{3}x^3 + \dots$$

$$= 1 + \frac{(1/2)}{1!}x + \frac{(1/2)(-1/2)}{2!}x^2 + \frac{(1/2)(-1/2)(-3/2)}{3!}x^3 + \dots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \dots$$

$$(1+x)^{1/3} = \binom{1/3}{0}x^0 + \binom{1/3}{1}x^1 + \binom{1/3}{2}x^2 + \binom{1/3}{3}x^3 + \dots$$

$$= 1 + \frac{(1/3)}{1!}x + \frac{(1/3)(-2/3)}{2!}x^2 + \frac{(1/3)(-2/3)(-5/3)}{3!}x^3 + \dots$$

$$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4 + \dots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

$$(1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4 - \frac{63}{256}x^5 + \frac{231}{1024}x^6 - \dots$$

Newton's binomial power expansions can be useful for numerically estimating square or cube roots. For instance,

$$\sqrt{26} = \sqrt{25+1} = \sqrt{25(1+\frac{1}{25})} = 5\left(1+\frac{1}{25}\right)^{1/2}$$

Using the first two terms of the series we get

$$5\left(1+\frac{1}{25}\right)^{1/2} = 5\left(1+\frac{1}{2}\left(\frac{1}{25}\right)\right) = 5.10$$

Or with the first three terms:

$$5\left(1+\frac{1}{25}\right)^{1/2} = 5\left(1+\frac{1}{2}\left(\frac{1}{25}\right) - \frac{1}{8}\left(\frac{1}{25}\right)^2\right) = 5.099$$

Other reading and references:
Calculus, 7E, James Stewart, Brooks/Cole Publishing, 2012
`https://en.wikibooks.org/wiki/LaTeX`
An Intro to LaTeX 2_ε