Harshad Numbers — Some Theory

Definition

A Harshad number is a positive integer that is divisible by the sum of its digits.

Question

Are there infinitely many Harshad numbers?

Can you prove it?

Theorem

There are infinitely many Harshad numbers.

Proof

$1, 10, 100, 1000, \ldots$ are all Harshad numbers. $\square$

Question

Are there infinitely many non-Harshad numbers?

Theorem

There are infinitely many non-Harshad numbers.

Proof

$11, 101, 1001, 10001, \ldots$ are all non-Harshad numbers. □

Proof

$19, 109, 1009, 10009, \ldots$ are all non-Harshad numbers. □

Consecutive Pairs

Notice that 20, 21 is a consecutive pair of Harshad numbers.

Are there infinitely many such pairs?

Theorem

There are infinitely many consecutive pairs of Harshad numbers.

Proof

$20, 21; 200, 201; 2000, 2001; 20000, 20001; \ldots.$ $\square$

Consecutive Triples

Are there any consecutive triples of Harshad numbers? Yes!

1, 2, 3

Are there any consecutive triples of Harshad numbers greater than 10? Yes!

110, 111, 112 is a consecutive triple of Harshad numbers.

Question

Do there exist infinitely many consecutive triples of Harshad numbers?

Question

Do there exist arbitrarily long finite sequences of consecutive Harshad numbers?

NO!

Theorem

There do not exist arbitrarily long finite sequences of consecutive Harshad numbers.

Proof

Let $\bar{a}$ represent a string of digits with digit sum s.
The digit sum of $\bar{a}01$ is $s + 1$.
The digit sum of $\bar{a}11$ is $s + 2$.
One of the two numbers $s + 1$ and $s + 2$ is even, but both of the numbers $\bar{a}01$ and $\bar{a}11$ are odd. So $\bar{a}01$ and $\bar{a}11$ cannot both be Harshad numbers. Hence no sequence of consecutive Harshad numbers contains two numbers of the form $\bar{a}01$ and $\bar{a}11$. □

Corollary

There is no sequence of more than 110 consecutive Harshad numbers.

Cooper & Kennedy's Theorem

Theorem (Cooper & Kennedy)

- *There does not exist a sequence of more than* 20 *consecutive Harshad numbers.*
- *There exists a sequence of exactly* 20 *consecutive Harshad numbers.*
- *There exist infinitely many sequences of exactly* 20 *consecutive Harshad numbers.*

Cooper & Kennedy

"...a lot of adjusting partial results, and a lot of luck and intuition."

Cooper and Kennedy found two numbers:
$\bar{a}$ with 1296 digits
$\bar{c}$ with 1298 digits
such that

$$\bar{a}_{3423103}00 \cdots 00\bar{c}$$

is the first number in a sequence of 20 consecutive Harshad numbers.

Theorem

There does not exist a sequence of more than 20 consecutive Harshad numbers.

We will prove this using three lemmas.

Notation

Again, let $\bar{a}$ be a (nonzero) string of digits.
And let s be the digit sum of $\bar{a}$.

Lemma 1

Suppose that

$$\bar{a}0, \bar{a}1, ..., \bar{a}9$$

is a sequence of ten consecutive Harshad numbers. Then 10 divides s*, the digit sum of* $\bar{a}$*.*

Proof

numbers:	$\bar{a}0$	$\bar{a}1$	...	$\bar{a}9$
digit sums:	s	$s+1$	...	$s+9$

Note: 10 must divide one of the digit sums.
Thus, 10 must divide the corresponding number.

The only number in the list that is divisible by 10 is $\bar{a}0$.
So 10 must divide s. □

Lemma 2

The numbers

$$\bar{a}00, \bar{a}01, \ldots, \bar{a}09, \bar{a}10$$

are not all Harshad numbers.

Proof

Suppose that they are.

By Lemma 1, 10 divides s.

numbers:	$\bar{a}00$	$\bar{a}01$	...	$\bar{a}09$	$\bar{a}10$
digit sums:	s	$s+1$	...	$s+9$	$s+1$

Then $s+1$ divides both of $\bar{a}01$ and $\bar{a}10$.
So, $s+1$ divides their difference: 9.
Thus $10 \leq s < s+1 \leq 9$. Contradiction. $\square$

Lemma 3

If $i \neq 9$, then

$$\bar{a}i9 \quad \textit{and} \quad \bar{a}(i+1)8$$

are not both Harshad numbers.

Proof

Suppose that they are.

The digit sum of each is $s + i + 9$.

Since the sum divides each of the numbers, it must divide their difference: 9.

So $s + i + 9 \leq 9$. Contradiction. □

Theorem

If $\bar{a}ij$ is the first number in a sequence of consecutive Harshad numbers of length greater than or equal to 20*, then* $i = 9$ *and* $j = 0$.

Proof

Suppose $i \neq 9$. Then we have Harshad numbers:
$\bar{a}ij, \ldots, \bar{a}i9$, $\bar{a}(i+1)0$, $\ldots, \bar{a}(i+1)8, \bar{a}(i+1)9$
By Lemma 3, these cannot both be Harshad numbers.
Contradiction. Thus $i = 9$.

Suppose $j \neq 0$. Then we have Harshad numbers:
$\bar{a}9j$, $\ldots, \bar{a}99$, $\bar{c}00, \ldots, \bar{c}09, \bar{c}10$
By Lemma 2, these cannot all be Harshad numbers.
Contradiction. Thus $j = 0$. □

Proof of Main Theorem

Theorem

There does not exist a sequence of more than 20 consecutive Harshad numbers.

Proof

Suppose there exists a sequence of 21 consecutive Harshad numbers, $x_1, x_2, \ldots, x_{21}$.

Then x_1 is the first term of $x_1, \ldots, x_{20}$.
By the previous theorem, the last digit of x_1 is 0.

Also, x_2 is the first term of $x_2, \ldots, x_{21}$.
By the previous theorem, the last digit of x_2 is 0.

But x_1 and x_2 are consecutive integers. Contradiction. □

Open Question

What is the smallest number initiating a sequence of 20 consecutive Harshad numbers?

Wilson's Theorem

Theorem (Wilson)

The smallest number initiating a sequence of twenty consecutive Harshad numbers

- *has at least* 1760 *digits;*
- *has at most* 1788 *digits;*
- *has a digit sum of* 15830*;*
- *ends with exactly* 1119 *nines followed by one zero.*

A Variation of Harshad Numbers

Definition

For $b \geq 2$, a b-Harshad number is a positive integer that is divisible by the sum of the digits in its base b expansion.

Recall Bases

$$327 = 3 \times 10^2 + 2 \times 10^1 + 7 \times 10^0$$

$$132_{(5)} = 1 \times 5^2 + 3 \times 5^1 + 2 \times 5^0 = 42$$

$$11010_{(2)} = 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 = 26$$

$$100_{(n)} = 1 \times n^2 + 0 \times n^1 + 0 \times n^0 = n^2$$

b-Harshad Numbers

Definition

For $b \geq 2$, a b-Harshad number is a positive integer that is divisible by the sum of the digits in its base b expansion.

Examples

$8 = 13_{(5)}$ is a 5-Harshad number. $1 + 3 = 4$ and $4 \mid 8$.

$6 = 110_{(2)}$ is a 2-Harshad number. $1 + 1 = 2$ and $2 \mid 6$.

$10000_{(b)}$ is a b-Harshad number for any b.

$1(b-1)_{(b)}$ is never a b-Harshad number.

Questions

- For each b, are there infinitely many b-Harshad numbers?
- For each b, are there infinitely many non-b-Harshad numbers?
- What about sequences of consecutive b-Harshad numbers?
- For each $b \geq 2$, are there pairs of consecutive b-Harshad numbers?
- Are there triples?
- For a given $b \geq 2$, is there a maximal number of consecutive b-Harshad numbers?
- Is the maximum the same for each b?