The Graph Brain Project

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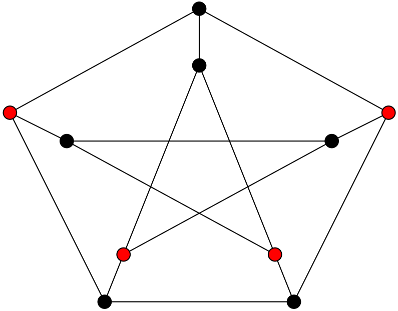
VCU Discrete Math Seminar
25 April 2018

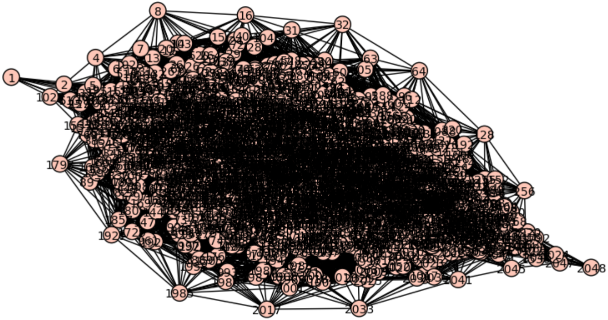
The Independence Number of a Graph



• The independence number α of a graph is the largest number of mutually non-adjacent vertices.

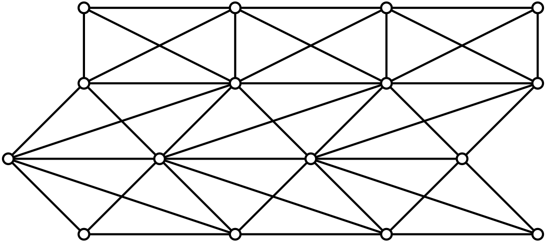
$$\alpha = 4.$$

The Independence Number of a Graph



Neil Sloane Challenge Problem Graph
order $n = 2048$.
$172 \leq \alpha \leq 174$.

An Application–Shannon Capacity



- The zero-error capacity of a alphabet is $\lim \sqrt[n]{\alpha(G^n)}$.

Computing the Independence Number

The Independent Set Decision Problem:

Given a graph G and an integer k, does G have an independent set of size at least k?

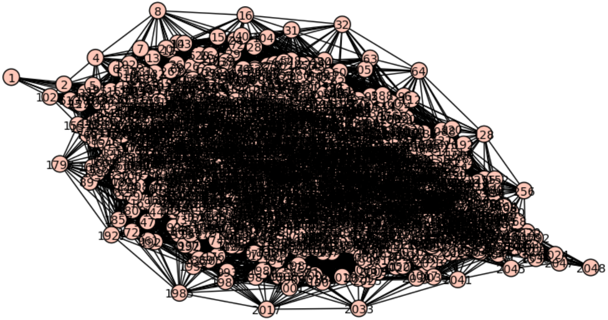
Independence number is NP-hard

Computing the Independence Number

Independence Number Theory

- Domination number: $\gamma \leq \alpha$
- Clique Covering number: $\alpha \leq \bar{\omega}$
- Chromatic number: $\alpha\chi \geq n$
- Matching number: $n - 2\mu \leq \alpha \leq n - \mu$.
- Clique Number: $\alpha(G) = \omega(\bar{G})$.
- Covering Number: $\alpha = n - \tau$.

Independence Number Theory



Neil Sloane Challenge Problem Graph
order $n = 2048$.
$172 \leq \alpha \leq 174$.

Observations about Bounds

- Bounds are functions of existing graph invariants.
- It is hard for humans to conjecture complex formulas.

Graffiti Bounds for the Independence Number

- $\alpha \geq$ radius.
- $\alpha \geq$ average distance.
- $\alpha \geq$ residue.
- $\alpha \geq$ `max_even_minus_even_horizontal`
- $\alpha \leq$ annihilation number.

Best Lower Bounds for Independence

- $\alpha \geq$ radius.
- $\alpha \geq$ residue.
- $\alpha \geq$ critical independence number
- $\alpha \geq$ `max_even_minus_even_horizontal`

Best Lower Bounds for Independence

Graph	Lower Bound	Value
Petersen	radius	2
	residue	3
	critical independence number	0
	max_even_minus_even_horizontal	1

Truth: $\alpha(\mathit{Petersen}) = 4$.

Best Upper Bounds for Independence

The Lovász number of a graph G is:

$$\vartheta(G) = \max[1 - \frac{\lambda_1(A)}{\lambda_n(A)}]$$

over all real matrices A with $a_{ij} = 0$ if $v_i \sim v_j$ in G, with eigenvalues $\lambda_1(A) \geq \ldots \geq \lambda_n(A)$

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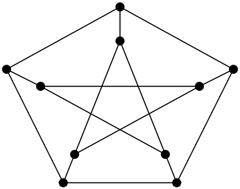


$\alpha \leq \vartheta = 4$

Best Upper Bounds for Independence

- $\alpha \leq$ annihilation number
- $\alpha \leq$ fractional independence number
- $\alpha \leq$ Lovász number
- $\alpha \leq$ Cvetković bound
- $\alpha \leq$ order - matching number.
- $\alpha \leq$ Hansen-Zheng bound.

(The *Hansen-Zheng bound* is
$\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + \text{order}^2 - \text{order} - 2 \cdot \text{size}} \rfloor$.)

Best Upper Bounds for Independence

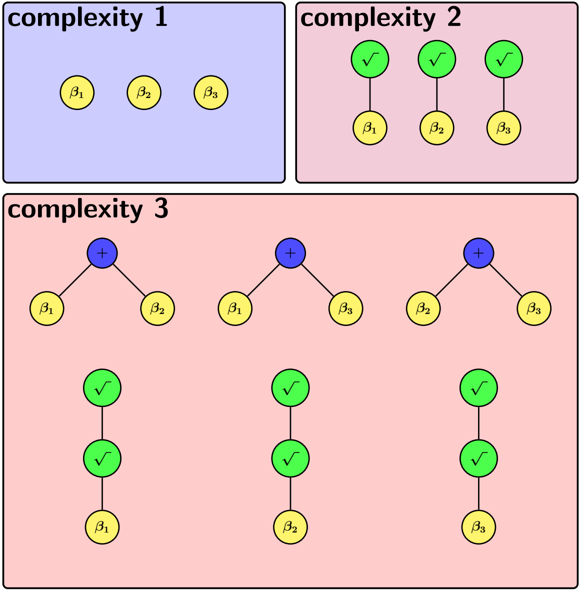
Graph	Upper Bound	Value
Petersen	`annihilation number`	5
	`fractional independence number`	5
	`Lovász number`	4
	`Cvetkovíc bound`	4
	`order - matching`	5
	`Hansen-Zheng bound`	8

Truth: $\alpha(Petersen) = 4$.

Finding New Bounds for the Independence Number?

- They need to be functions of existing invariants.
- They must be true for all graphs.
- So they must be true for all common graphs (and all published graphs).

Computer Methods to Find New Bounds for α



Computer Methods to Find New Bounds for α

Hao Wang

Computer Methods to Find New Bounds for α

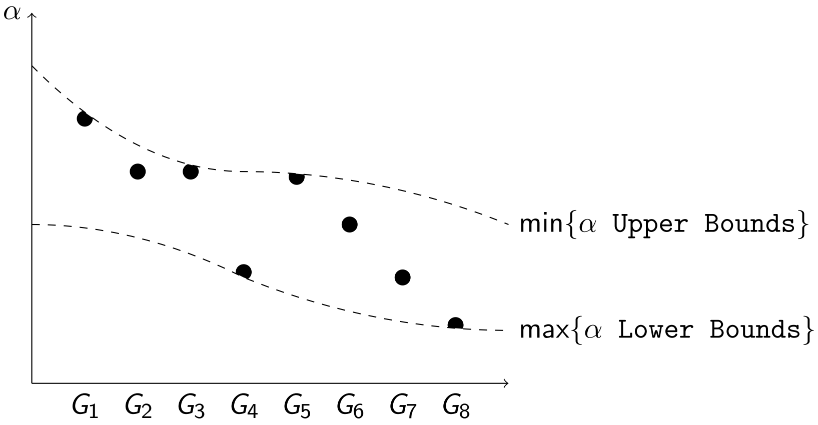
- Generating expressions isn't enough.
- They need to be filtered somehow.
- Truth for examples is one filter.

Computer Methods to Find New Bounds for α

Fajtlowicz's Dalmatian heuristic:

only store a statement if it gives a better bound for at least one stored object.

Computer Methods to Find New Bounds for α



Our CONJECTURING Program

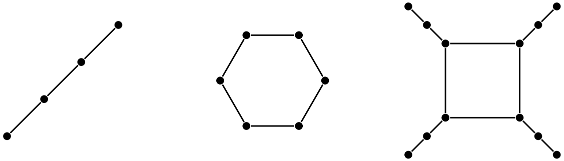
Inputs:

- Examples of graphs. (520)
- Definitions of invariants for these objects. (159)
- An Invariant you want bounds for.
- Whether you want upper or lower bounds.

Two Conjectured Theorems

Theorem

For any connected graph, $\alpha \leq$ *order* $-$ *radius.*



r-ciliates: $C_{1,1}$, $C_{3,0}$, $C_{2,2}$

Two Conjectured Theorems

Theorem

For any connected graph, $\alpha \leq$ *order* $-$ *radius.*

Proof.

Let G be a connected graph with radius r, and r-ciliate $C_{p,q}$ (with $r = p + q$). Note that an r-ciliate is bipartite. It is easy to check that $n(C_{p,q}) = 2p(q+1)$, $\alpha(C_{p,q}) = p(q+1)$, and $\alpha(C_{p,q}) \leq n(C_{p,q}) - r(C_{p,q})$.
Let $V' = V(G) \setminus V(C_{p,q})$, and $n' = |V'|$. Then $\alpha(G) \leq \alpha(C_{p,q}) + n' \leq (n(C_{p,q}) - r(C_{p,q})) + n' = (n(G) - n') - r(G) + n' = n(G) - r(G)$. $\square$

Two Conjectured Theorems

Theorem

For any graph G, $\alpha(G) \geq \Delta(G) - T(G)$.

$\Delta(G) =$ maximum degree, $T(G) =$ number of triangles.

Proof.

Assume the statement is true for graphs with fewer than m edges. Let G be a graph with m edges and v be a vertex of maximum degree. It is easy to see that the conjecture is true in any case where $T(G) = 0$. We can assume there is an edge e not incident to v in some triangle. Let G' be the graph formed by removing edge e (but not its incident vertices). So, by assumption, $\alpha(G') \geq \Delta(G') - T(G')$. We see that $\alpha(G') - 1 \leq \alpha(G)$, $\Delta(G') = \Delta(G)$ and that $T(G') + 1 \leq T(G)$. Then $\alpha(G) \geq \alpha(G') - 1 \geq (\Delta(G') - T(G')) - 1 \geq \Delta(G) - (T(G) - 1) - 1 = \Delta(G) - T(G)$.

□

What else would be Useful?

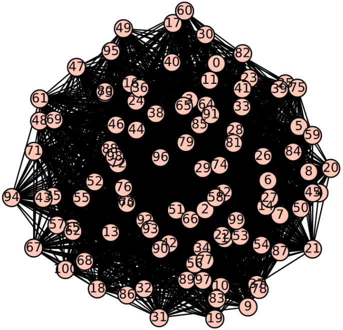
- We might want conjectures that are not implied by existing theory,
- that is, conjectures that give a better bound for at least one graph,
- so, for us, at least one graph in our database.
- We call this the `theory` input.

Three New Conjectures

- $\alpha \geq$ `min(girth, floor(lovasz_theta))`
- $\alpha \geq$ `ceil(lovasz_theta) - girth`
- $\alpha \leq$ `(average_distance)^(degree_sum)`

Three New Conjectures

(2) $\alpha \geq$ `ceil(lovasz_theta) - girth`



Paley graph on the field of order 101

$\alpha = 5$

girth=3

lovasz_theta=10.049876

(Jianxiang Chen)

Three New Conjectures

$\alpha \leq$ (average_distance)^(degree_sum)

- True? (Chen proof sketch).
- Tested on all graphs of order ≤ 10.
- Tested on Random Graphs of all orders up to order 120.

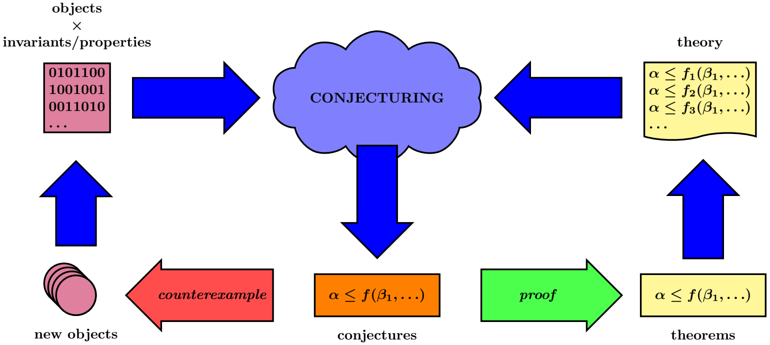
Three New Conjectures

(1) $\alpha \geq$ `min(girth, floor(lovasz_theta))`

Equivalently, $\alpha \geq$ `girth` or $\alpha =$ `floor(lovasz_theta)`

The best one!

Our CONJECTURING program. Iterate to the truth.

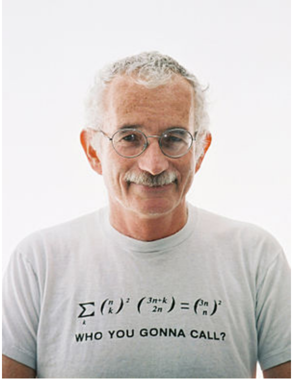


The Graph Brain Project

- Add every published graph, invariant and independence number theorem and operator.
- Humans couldn't conjecture a simpler bound, true of all published graphs.
- We might also be able to say concrete things about bounds - maybe no expression with complexity 11 or less predicts the independence number of a certain graph.

The Graph Brain Project

Computers Won't Take Over Mathematics

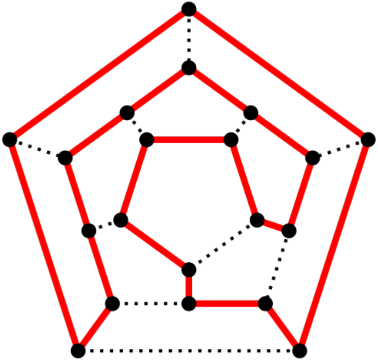


Doron Zeilberger

These are new computer tools—to help us achieve our mathematical goals.

Mathematics still depends on our human goals.

The Properties Version of CONJECTURING



A graph is Hamiltonian is there is a cycle containing all the vertices of the graph.

The Properties Version of CONJECTURING

- If a graph is a clique then it is Hamiltonian. (T)
- If a graph is connected and Dirac then it is Hamiltonian. (T)
- If a graph is eulerian, regular and 2-connected then it is Hamiltonian. (F)

The Properties Version of CONJECTURING

- If a graph is a clique then it is Hamiltonian. (T)
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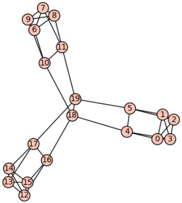
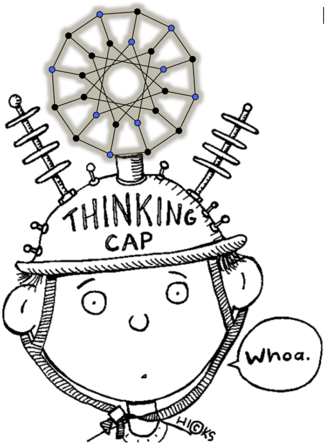


- If a graph is eulerian and `has_radius_equal_diameter`. (O)

Summer 2018 Project

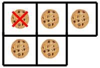


The General Applicability of CONJECTURING

Need Inputs:

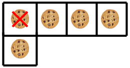
- Objects,
- Invariants,
- Properties,
- Theorems.

An Example: Chomp!

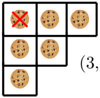


(3,2)

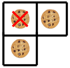
(1)



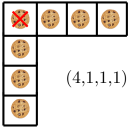
(4,1)



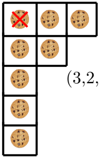
(3,2,1)



(2,1)



(4,1,1,1)



(3,2,1,1,1)

An Example: Chomp!

Conjectured Theorem:

For any position where the *previous-player-to-play* has a winning strategy (a P-position),

the number of cookies on the board $\geq$ the number of (non-empty) columns -1.

Thank You!

Automated Conjecturing in Sage:
nvcleemp.github.io/conjecturing/

Graph Brain Project:
github.com/math1um/objects-invariants-properties

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