6
Hamiltonian Graphs

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1. Introduction

If G is a graph, a **Hamiltonian circuit** in G is a circuit which contains every vertex of G, and a **Hamiltonian path** in G is a path which contains every vertex of G. So if G has order p, then a Hamiltonian circuit has length p and a Hamiltonian path has length $p-1$. (Throughout this chapter we shall assume, usually without saying so, that $p \geqslant 3$.) A graph which contains a Hamiltonian circuit is called a **Hamiltonian graph**, and a graph which contains a Hamiltonian path is called a **traceable graph**.

Hamiltonian graphs are named after William Rowan Hamilton, although they were studied earlier by Kirkman. In 1856, Hamilton invented a mathematical game, the "icosian game", consisting of a dodecahedron each of whose twenty vertices was labeled with the name of a city. The object of the game was to travel along the edges of the dodecahedron, visiting each city exactly once and returning to the initial point. In graph-theoretical terms, the aim is to find a Hamiltonian circuit in the dodecahedral graph (see Fig. 1, where such a circuit is indicated by heavy lines). For a picture of the icosian game, and more details concerning the origin of this problem, see [**10**, Chapter 2].

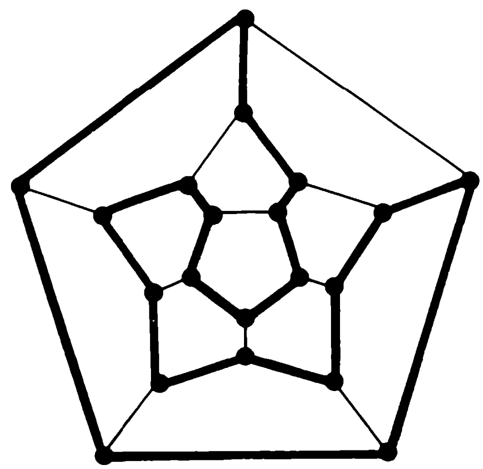


Fig. 1

Another example concerns a "squirrel cage", which consists of a $3 \times 3 \times 3$ cube made up from 27 small cubes. Is it possible for a child to start at one corner and visit each cube in turn, ending at the central cube? If we consider the graph whose vertices represent the small cubes, and whose edges join those pairs of vertices which correspond to adjacent cubes, then the problem asks whether this graph has a Hamiltonian path with given initial and terminal vertices—the answer will be given in Section 3. Many other problems can also be expressed in terms of Hamiltonian paths and circuits, such as the knight's-tour problem on a chessboard, and the "Tower of Hanoi"; the reader is referred to **[6]**, **[10]**, or **[19]** for further details.

It is clear that the complete graph K_p is Hamiltonian, since we can start at any vertex and go successively to any other vertex not yet visited. However, if we "weight" the edges of K_p, then the problem of finding a minimum weight Hamiltonian circuit is a difficult one. It is usually called the "traveling salesman problem", and represents the problem of finding how a traveling salesman can visit each of p towns exactly once in the shortest possible time. More precisely, the aim is to find a "good" or "efficient" algorithm which produces the required Hamiltonian circuit, but although there is an extensive literature on this problem, no efficient algorithm is known. However, this is primarily an integer programming problem, rather than a problem on Hamiltonian graphs, and so we shall not consider it here (see, for example, **[4]**, or **[20**, Chapter 10**]**).

Another example we shall be considering is the Petersen graph (see Fig. 2). It is not difficult to check that it is a non-Hamiltonian graph which is traceable. (In fact, it is "homogeneously traceable", in the sense that it contains a Hamiltonian path starting at any given vertex.) The Petersen graph forms the basis of most of the counter-examples to conjectures on Hamiltonian graphs. We shall discuss further non-Hamiltonian graphs in Sections 3 and 6.

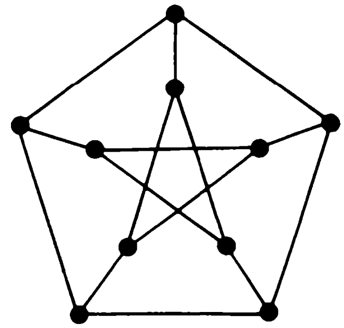


Fig. 2

Unlike the situation for Eulerian graphs, there is no known non-trivial characterization of Hamiltonian graphs, although several necessary conditions (see Section 3) and several sufficient conditions (see Sections 4 and 5) have been discovered. In fact, the problem of determining which graphs are Hamiltonian is one of the major unsolved problems of graph theory. As a consequence, a very large literature exists on the subject, and there are several interesting expository accounts. The interested reader is referred in particular to the surveys of Berge [**6**, Chapter 10], Bondy and Murty [**17**, Chapters 4 and 9], Nash-Williams [**49**], [**50**], [**52**], Chartrand, Kapoor and Kronk [**19**], Chvátal [**21**] and Lesniak [**43**]. In this chapter, we shall often find it convenient to refer to these articles, rather than to the original papers.

Finally, we should like to thank all those who helped in the preparation of this chapter—in particular, J. A. Bondy, M. Chein and C. Thomassen.

2. A Simple Theorem

In this section we shall show that the problems of determining which graphs, digraphs and bipartite graphs are Hamiltonian or traceable are essentially the same. In this context, a digraph D is a **Hamiltonian digraph** if it contains a directed circuit which passes through every vertex of D, and is a **traceable digraph** if it contains a directed path which passes through every vertex of D. For example, consider the digraph whose vertices represent jobs to be done, and where there is an arc from vertex i to vertex j if the work for job i can be done before that for job j; if this digraph is traceable, then the jobs can be done in order, and the determination of such an ordering optimal under certain conditions is an important problem in operations research.

We can now state and prove the main result of this section, first proved by Nash-Williams [**49**]:

Theorem 2.1. *The following problems are equivalent*:

(*i*) *the determination of all Hamiltonian graphs*;

(*ii*) *the determination of all traceable graphs*;

(*iii*) *the determination of all Hamiltonian digraphs*;

(*iv*) *the determination of all traceable digraphs*;

(*v*) *the determination of all Hamiltonian bipartite graphs.*

Proof. (*i*) ⇔ (*ii*). Let G be a graph, and let H be the graph obtained by taking a new vertex and joining it to every vertex of G; then G is traceable if and only if H is Hamiltonian, and so if we know which graphs are Hamiltonian, we can determine which graphs are traceable. (Note that most of the known conditions for a graph to be traceable are obtained in this way from conditions for a graph to be Hamiltonian.) Conversely, let H be a graph, let v be a vertex of H, and let G be the graph obtained by taking three new vertices x, y and z, joining z to all the neighbors of v, and adding the edges vx and yz; then H is Hamiltonian if and only if G is traceable, and so if we know which graphs are traceable, we can determine which graphs are Hamiltonian. So (*i*) ⇔ (*ii*).

(*iii*) ⇔ (*iv*). This is very similar to the above proof.

(*i*) ⇔ (*iii*). Let G be a graph, and let D be the digraph obtained by replacing each edge of G by two opposite arcs; then G is a Hamiltonian graph if and only if D is a Hamiltonian digraph, and so if we know which symmetric digraphs are Hamiltonian, we can determine which graphs are Hamiltonian. Conversely, let D be a digraph, and let G be a graph constructed from D in the following manner: to each vertex v of D associate a path of length three with initial and terminal vertices a_v and b_v, and choose these paths to be vertex-disjoint; let G consist of these paths, together with the edges $b_v a_w$ for every arc vw of D. Then it is easily shown that D is a Hamiltonian digraph if and only if G is a Hamiltonian graph, and so if we know which graphs are Hamiltonian, we can determine which digraphs are Hamiltonian. So (*i*) ⇔ (*iii*).

(*i*) ⇔ (*v*). The graph G in the preceding part is bipartite, so that (*iii*) ⇒ (*v*). But problem (*v*) is a special case of problem (*i*), and so (*i*) ⇔ (*v*). ‖

Note that the problem of recognizing Hamiltonian graphs is NP-complete. This has been established by Karp, Lawler and Tarjan (see [**1**, Chapter 10]).

3. Some Necessary Conditions

We now look at some necessary conditions for Hamiltonian graphs. Further details can be found in Chvátal's survey [**21**], or in his articles [**22**], [**23**]. We start with the following simple, but important, theorem:

Theorem 3.1. *Let G be a Hamiltonian graph, let S be a non-empty proper subset of the vertex-set $V(G)$, and let $c(G-S)$ be the number of components of the graph $G-S$. Then*

$$c(G-S) \leqslant |S|.$$

Proof. Let C be a Hamiltonian circuit of G. Then for every non-empty proper subset S of $V(G)$ we have $c(C-S) \leqslant |S|$. But $C-S$ is a spanning subgraph of $G-S$, and so $c(G-S) \leqslant c(C-S)$. The result follows. ||

If we consider only sets S of cardinality 1, we obtain the following corollary:

Corollary 3.2. *Every Hamiltonian graph is 2-connected.* ||

Chvátal has defined a graph to be **1-tough** if $c(G-S) \leqslant |S|$ for every non-empty proper subset S of $V(G)$. It follows from Theorem 3.1 that every Hamiltonian graph is 1-tough.

The converse of Corollary 3.2 is not true. For example, if $G = K_{r,s}$, where $r < s$, then G is r-connected but not 1-tough, and therefore not Hamiltonian. Using this we can deduce that the squirrel-cage problem of Section 1 is impossible. Since the "squirrel-cage graph" is easily seen to be bipartite, every Hamiltonian path starting at a corner and ending at the center induces a Hamiltonian circuit in $K_{13,14}$ (on adding one extra edge joining the starting cube and the center cube), giving the required contradiction.

The converse of Theorem 3.1 is also false. For example, the Petersen graph is a 1-tough graph which is not Hamiltonian. The smallest such graph is shown in Fig. 3.

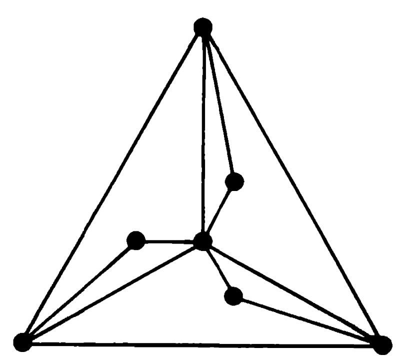


Fig. 3

In [22], Chvátal extended these ideas by defining the **toughness** $t(G)$ of a graph G (other than a complete graph) by

$$t(G) = \min \frac{|S|}{c(G-S)},$$

where the minimum extends over all separating sets S of G—that is, all subsets S of $V(G)$ for which $c(G-S) > 1$. So G is 1-tough if $t(G) \geqslant 1$. If we know the connectivity κ and the independence number α of G, we can easily obtain a lower bound for $t(G)$; note that this bound is exact for the complete bipartite graphs $K_{r,s}$:

Theorem 3.3. *Let G be a graph with connectivity κ and independence number α. Then $t(G) \geqslant \kappa/\alpha$.*

Proof. If S is a separating set of vertices of G, then $|S| \geqslant \kappa$. Also it is clear that $c(G-S) \leqslant \alpha$. The result follows. ||

In **[22]** Chvátal obtained various other results relating the toughness of G to other parameters. He also conjectured the existence of a number t_0 such that every graph with toughness $t \geqslant t_0$ is necessarily Hamiltonian. If this conjecture is true for $t_0 = 2$, then we can deduce Fleischner's theorem on the squares of blocks (see Theorem 5.6), since $t(G^2) \geqslant \kappa$, by a result of **[22]**. Chvátal also conjectured that every graph with toughness $t > \frac{3}{2}$ is Hamiltonian, but this was disproved by Thomassen, who gave the following counter-example: take a trivalent 3-connected graph with no Hamiltonian path (such graphs exist—see Theorem 6.4(*iii*)), and replace every vertex by a triangle; then the resulting graph H can be shown to have toughness $\frac{3}{2}$ (see **[22]** for a proof). If we now let G be the graph obtained by adding a new vertex and joining it to every vertex of H, then $t(G) > \frac{3}{2}$ and G is non-Hamiltonian.

Another conjecture of Chvátal was proved by Jung **[40]**:

Theorem 3.4. *If G is 1-tough, then either G is Hamiltonian, or its complement $\bar{G}$ contains the graph of Fig. 4 as a subgraph.* ||

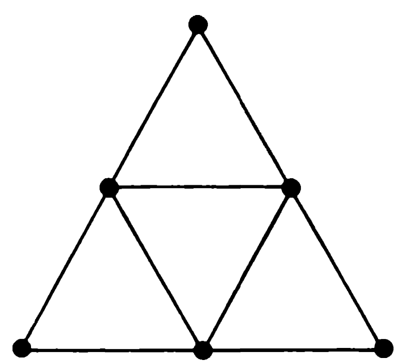


Fig. 4

We now turn our attention to "weakly Hamiltonian graphs" introduced by Chvátal **[23]**. If G is any graph with vertex-set V and edge-set E, and if C is any Hamiltonian circuit in G, then the **characteristic function** of C is the function $f: E \rightarrow \{0, 1\}$, defined by $f(e) = 1$ if $e \in C$, and $f(e) = 0$ otherwise. For any $v \in V$, we define $f(v)$ to be the sum of the values of $f(e)$, taken over all edges incident to v, and for any $T \subseteq V$, we define $f[T]$ to be the sum of the values of $f(e)$, taken over all edges e with both incident vertices in T. We can now prove the following simple results:

$$f(v) \leqslant 2, \quad \text{for all } v \in V; \qquad f[T] \leqslant |T|-1, \quad \text{for all non-empty proper subsets } T \text{ of } V. \tag{1}$$

Conversely, let $f: E \to [0, \infty)$ be an integer-valued function satisfying both (1) and

$$f[V] = p, \tag{2}$$

where p is the order of G. Then f is the characteristic function of a Hamiltonian circuit of G.

In [23], Chvátal proved that if f is an integer-valued function satisfying (1), then f must satisfy what he called a "comb inequality". Let $W_0, \ldots, W_n$ be non-empty proper subsets of V satisfying $|W_i \cap W_0| = 1$ for $i = 1, \ldots, n$, and let $K = W_0 \cup W_1 \cup \ldots \cup W_n$. Then

$$f_K \leqslant |W_0| + \sum_{i=1}^{n} |W_i - 1| - [\tfrac{1}{2}(n+1)], \tag{3}$$

where f_K is the sum of the values of $f(e)$, taken over all edges e with both incident vertices in the same W_i.

Chvátal defined a graph to be **weakly Hamiltonian** if there exists a function $f: E \to [0, \infty)$ (not necessarily integer-valued) satisfying (1), (2) and (3). It follows that every Hamiltonian graph is weakly Hamiltonian. But the duality theorem of linear programming can be used to give a characterization of weakly Hamiltonian graphs, which in turn gives a necessary condition for a graph to be Hamiltonian. A weaker, but more easily stated, version of this condition is the following:

Theorem 3.5. *If $G = (V, E)$ is a weakly Hamiltonian graph, then there is no partition $V = R \cup S \cup T$ into pairwise disjoint (possibly empty) sets with $T \neq V$, and*

$$|S| + \sum [\tfrac{1}{2}m(C, T)] < c(T),$$

where the summation extends over all components C of the subgraph induced by R, $m(C, T)$ is the number of edges joining C to T, and $c(T)$ is the number of components of the subgraph induced by T. ‖

The smallest weakly Hamiltonian graph which is not Hamiltonian is the Petersen graph. To see that it is weakly Hamiltonian, let $f(e) = \frac{2}{3}$ for each edge e. Some idea of how closely weakly Hamiltonian graphs approximate Hamiltonian graphs is given by the following theorem, also due to Chvátal [23]:

Theorem 3.6. *Every weakly Hamiltonian graph is 1-tough, has a 2-factor, and contains a circuit passing through any three given vertices.* ‖

4. Some Sufficient Conditions on Valencies

Most of the known sufficient conditions for a graph G to be Hamiltonian assert that if G is "large enough", or if G "has enough edges", then G is Hamiltonian. We shall essentially follow the method of Bondy and Chvátal **[16]**, which unifies most of the known conditions and can also be applied to other problems in graph theory. We begin with a lemma whose proof contains the basic idea of the method:

Lemma 4.1. *Let G be a graph of order p, and let v and w be two non-adjacent vertices whose valencies satisfy*

$$\rho(v)+\rho(w) \geqslant p.$$

Then G is Hamiltonian if and only if $G+vw$ is Hamiltonian.

Proof. If G is Hamiltonian, then $G+vw$ is clearly Hamiltonian also. Conversely, suppose that $G+vw$ is Hamiltonian, but that G is not. Then G contains a Hamiltonian path from v to w—say, $v_1(=v), v_2, v_3, \ldots, v_{p-1}, v_p(=w)$. Let $S = \{v_k : vv_{k+1} \in E\}$ and $T = \{v_k : v_k w \in E\}$; then, by hypothesis,

$$|S|+|T| = \rho(v)+\rho(w) \geqslant p.$$

But $|S \cup T| < p$ since $w \notin S \cup T$, and so there exists k such that v is adjacent to v_{k+1} and w is adjacent to v_k. So G contains the Hamiltonian circuit

$$v, v_{k+1}, v_{k+2}, \ldots, v_{p-1}, w, v_k, v_{k-1}, \ldots, v_2, v$$

(see Fig. 5), giving the required contradiction. ‖

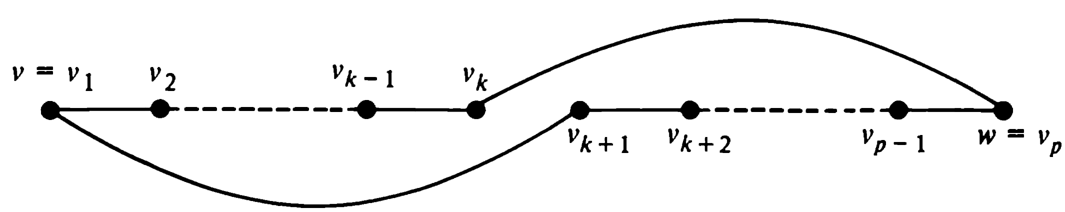


Fig. 5

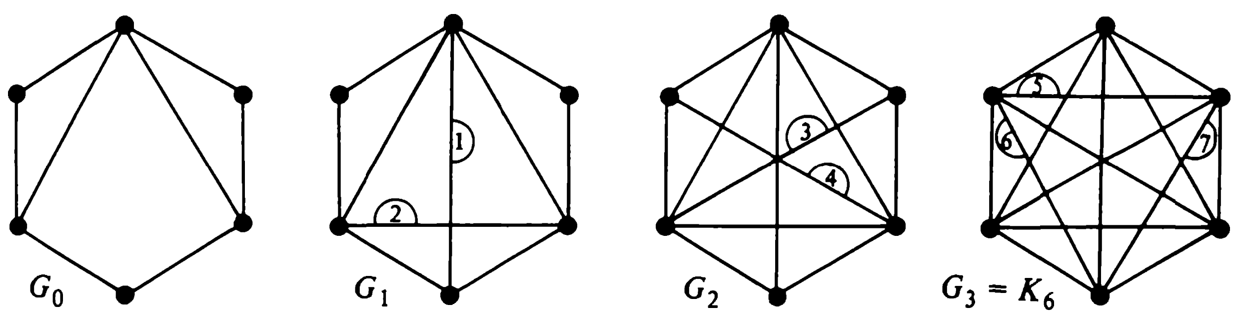
As corollaries we obtain the following theorems of Ore (1960) and Dirac (1952):

Corollary 4.2 (Ore's Theorem). *Let G be a graph of order p ($\geqslant$3). If $\rho(v)+\rho(w) \geqslant p$ for each pair of non-adjacent vertices v and w in G, then G is Hamiltonian.* ‖

Corollary 4.3 (Dirac's Theorem). *Let G be a graph of order p ($\geqslant 3$). If $\rho(v) \geqslant \frac{1}{2}p$ for each vertex v in G, then G is Hamiltonian.* ‖

(In fact, Nash-Williams (1969) has proved that under the conditions of Dirac's theorem, G contains at least $[\frac{5}{224}(n+10)]$ edge-disjoint Hamiltonian circuits. Jung **[41]** has also proved that if G is 1-tough, and if for each pair of non-adjacent vertices v and w in G, $\rho(v)+\rho(w) \geqslant p-4$, where $p \geqslant 11$, then G is Hamiltonian.)

Motivated by the result of Lemma 4.1, Bondy and Chvátal defined the **closure** cl(G) of a graph G to be the smallest graph H such that (i) G is a spanning subgraph of H, and (ii) $\rho_H(v)+\rho_H(w) < p$ for every pair of non-adjacent vertices v and w in H. (In [16], cl(G) is called the "p-closure".) The closure cl(G) can be obtained from G by the recursive procedure of joining two vertices whenever the sum of their valencies is at least p—this recursive procedure can be effected in $O(p^4)$ steps. For example, the closure of the graph G_0 in Fig. 6(a) is K_6, as can be seen by successively adding edges between non-adjacent vertices with valency-sum 6 or more. The various steps are shown in Fig. 6(b), (c) and (d).



(a) (b) (c) (d)

Fig. 6

The importance of the closure operation lies in the following theorem and corollary, due to Bondy and Chvátal **[16]**:

Theorem 4.4. *G is Hamiltonian if and only if* cl(G) *is Hamiltonian.*

Proof. The result follows by applying Lemma 4.1 each time an edge is added to form the closure. ‖

Corollary 4.5. *If* cl(G) *is a complete graph, then G is Hamiltonian.* ‖

This corollary incorporates most of the known conditions on the valencies of Hamiltonian graphs. Before stating these conditions we outline an algorithm in $O(p^3)$ steps for finding a Hamiltonian circuit in G from a Hamiltonian circuit in cl(G). We first assign to each added edge vw of cl(G) a label $\alpha(vw)$

corresponding to the order in which it was added in the above recursive procedure (see Fig. 6). Now let C be a Hamiltonian circuit in cl(G). If every edge of C lies in G, then we have a Hamiltonian circuit in G. If not, we choose the edge vw of C with the highest label. By the proof of Lemma 4.1, there exist two consecutive vertices v_k and v_{k+1} of C such that v is adjacent to v_{k+1} and v_k is adjacent to w, and such that $\alpha(vv_{k+1}) < \alpha(vw)$ and $\alpha(wv_k) < \alpha(vw)$—such vertices can be found in $O(p)$ steps. By deleting from C the edges v_kv_{k+1}, and adding the edges vv_{k+1} and wv_k, we obtain a Hamiltonian circuit in cl(G) all of those labeled edges have label less than $\alpha(vw)$. By repeating this procedure at most $O(p^2)$ times, we obtain a Hamiltonian circuit in G.

We can now use Corollary 4.5 on the closure of G to prove a result of Las Vergnas (1971):

Theorem 4.6. *Let G be a graph with vertex-set $\{v_1, \ldots, v_p\}$, and suppose that there do not exist integers i, j satisfying*

$$\begin{cases} i < j,\ i+j \geqslant p,\ v_iv_j \notin E,\ \rho(v_i) \leqslant i,\ \rho(v_j) \leqslant j-1, \\ \text{and } \rho(v_i)+\rho(v_j) \leqslant p-1. \end{cases} \tag{4}$$

Then G is Hamiltonian.

Proof. By Corollary 4.5 we need only prove that cl(G) is a complete graph. So suppose that $H = $ cl(G) is not a complete graph, and choose in H two non-adjacent vertices v_i and v_j such that

(*i*) j is as large as possible, and

(*ii*) i is as large as possible subject to condition (*i*).

We shall obtain a contradiction by showing that i and j satisfy all of the properties (4) in the statement of the theorem.

First of all, $i < j$, by (*i*). Next, since H is the closure, we have

$$\rho_H(v_i)+\rho_H(v_j) \leqslant p-1 \tag{5}$$

(where ρ_H denotes the valency in H), and so $\rho(v_i)+\rho(v_j) \leqslant p-1$. By (*i*), v_i must be adjacent in H to all those v_k with $k > j$, and so

$$\rho_H(v_i) \geqslant p-j. \tag{6}$$

By (*ii*), v_j must be adjacent in H to all those v_k with $k > i$, $k \neq j$, and so

$$\rho_H(v_j) \geqslant p-i-1. \tag{7}$$

Now (5) and (6) imply that

$$\rho(v_j) \leqslant \rho_H(v_j) \leqslant p-1-(p-j) = j-1,$$

and (5) and (7) imply that

$$\rho(v_i) \leqslant \rho_H(v_i) \leqslant p-1-(p-i-1) = i.$$

Finally, (6) and (7) imply that

$$\begin{aligned} i+j &\geqslant 2p-1-\rho_H(v_i)-\rho_H(v_j), \\ &\geqslant p, \text{ by (5).} \ \| \end{aligned}$$

The graph G of Fig. 6(a) shows that Theorem 4.4 is stronger than Theorem 4.6, but Theorem 4.6 is stronger than the following result of Chvátal (1972) which involves only the valencies. The deduction of Chvátal's theorem from Las Vergnas' theorem is not immediate (see, for example, [**16**, Appendix 2]); alternatively, Chvátal's result can be proved by imitating the proof of Las Vergnas' theorem.

Theorem 4.7 (Chvátal's Theorem). *Let G be a graph with non-decreasing valency-sequence $\rho_1, \rho_2, \ldots, \rho_p$. If $\rho_k \leqslant k < \frac{1}{2}p \Rightarrow \rho_{p-k} \geqslant p-k$, for each k, then G is Hamiltonian.* ‖

Note that Chvátal's theorem (and hence Las Vergnas' theorem) generalizes the earlier results of Ore and Dirac (Corollaries 4.2 and 4.3), as well as other results of Pósa, Bondy and Nash-Williams. However, if we consider only conditions on the valencies, then Chvátal's theorem is in some sense the best possible result. To see this, we consider the graphs $G(r, p)$ defined as follows: if $1 \leqslant r < \frac{1}{2}p$, let $G(r, p)$ be the graph of order p with vertex-set $S \cup T \cup U$, where $|S| = |T| = r$ and $|U| = p-2r$, and where two vertices are joined if either of them belongs to S, or if both of them belong to U (see Fig. 7). Note that the subgraphs induced by S, T and U are K_r, $\bar{K}_r$ and K_{p-2r}, respectively.

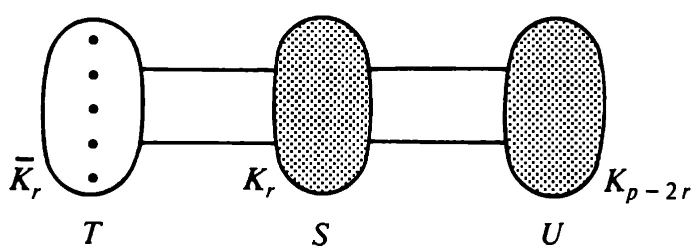


Fig. 7

Also, if $v \in S$ then $\rho(v) = p-1$, if $v \in T$ then $\rho(v) = r$, and if $v \in U$ then $\rho(v) = p-r-1$; it follows that the (non-decreasing) valency-sequence of $G(r, p)$ is

$$\underbrace{r, r, \ldots, r,}_{r \text{ terms}} \quad \underbrace{p-r-1, p-r-1, \ldots, p-r-1,}_{p-2r \text{ terms}} \quad \underbrace{p-1, p-1, \ldots, p-1.}_{r \text{ terms}}$$

It can be shown that $G(r,p)$ is the only graph with this valency-sequence. Moreover, $G(r, p)$ is not Hamiltonian since $c(G-S) = p+1 > |S|$, and hence $G(r, p)$ is not 1-tough.

A sequence $\rho_1, \rho_2, \ldots, \rho_p$ is said to be **majorized** by a sequence $\rho'_1, \rho'_2, \ldots, \rho'_p$ if $\rho_i \leqslant \rho'_i$ for $i = 1, 2, \ldots, p$. A graph G is **valency-majorized** by a graph H if the non-decreasing valency-sequence of G is majorized by the non-decreasing valency-sequence of H. (Note that G and H must have the same order.) We can now prove the following result of Chvátal (1972):

Theorem 4.8. *If G is a non-Hamiltonian graph of order p, then G is valency-majorized by some G(r, p).*

Proof. Let $\rho_1 \leqslant \rho_2 \leqslant \ldots \leqslant \rho_p$ be the non-decreasing valency sequence of G. By Chvátal's theorem (Theorem 4.7), there is an integer $k < \frac{1}{2}p$ such that $\rho_k \leqslant k$ and $\rho_{p-k} < p-k$, and so G is valency-majorized by $G(k, p)$. ‖

There are also various theorems which relate to the number of edges in the graph. One of these is the following result, due to Ore and Bondy:

Theorem 4.9. *If G is a graph with p vertices and more than* $\binom{p-1}{2}+1$ *edges, then G is Hamiltonian. Furthermore, the only non-Hamiltonian graphs with p vertices and exactly* $\binom{p-1}{2}+1$ *edges are the graphs* $G(1, p)$ *and (for* $p = 5$*)* $G(2, 5)$ (see Fig. 8).

Proof. By Theorem 4.8, every non-Hamiltonian graph of order p can be valency-majorized by $G(r, p)$, for some r, and so the number of edges of G is at most $|E(G(r, p))|$, which is $\binom{p-r}{2}+r^2$. But an easy calculation shows that

$$\binom{p-r}{2}+r^2 \leqslant \binom{p-1}{2}+1,$$

with equality if $r = 1$, or if $r = 2$ and $p = 5$. (Recall that $G(r, p)$ is the only graph with its valency-sequence.) The result follows. ‖

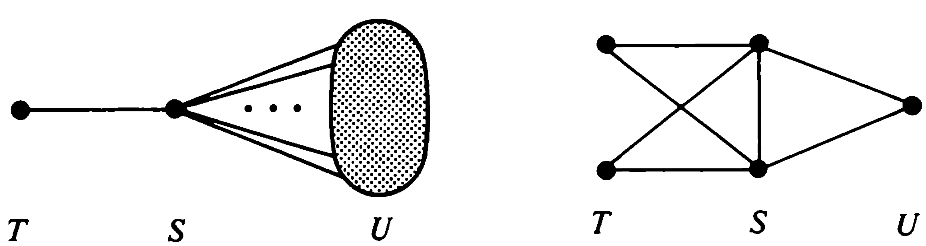


Fig. 8

Another result on the number of edges of a Hamiltonian graph is of a probabilistic nature, and is due to Komlós and Szemerédi (1978)—private communication of Chvátal—and improved an earlier result of Pósa (1976).

Theorem 4.10. *Suppose that we are given p vertices, and that we place* $[\frac{1}{2}p \log p + \frac{1}{2}p \log \log p] + O(p)$ *edges between them at random. If* $P(p, c)$ *is the probability that the resulting graph is Hamiltonian, then for all sufficiently large c,*

$$\lim_{p \to \infty} P(p, c) = 1. \parallel$$

There are some valency sequences which do not satisfy the hypothesis of Chvátal's theorem (Theorem 4.7), but which are necessarily the valency-sequences of Hamiltonian graphs. A valency-sequence is called **forcibly Hamiltonian** if every graph with this valency-sequence is Hamiltonian. Some of these sequences have been characterized by Nash-Williams [**51**], who proved in particular that every regular k-valent graph of order $2k+1$ is Hamiltonian. Further results on regular graphs have been obtained by Bollobás, Erdős, and Hobbs, and recently by Jackson [**37**], who obtained the following result:

Theorem 4.11. *Let G be a regular 2-connected graph of order p and valency* ρ, *where* $\rho \geqslant \frac{1}{3}p$. *Then G is Hamiltonian.* $\parallel$

This result is best possible when $\rho = 3$, because of the Petersen graph, and is almost best possible when $\rho \geqslant 4$, since there exist infinite families of regular ρ-valent 2-connected non-Hamiltonian graphs of order $3\rho+4$ (if ρ is even) or $3\rho+5$ (if ρ is odd). Häggkvist has made an analogous conjecture for bipartite graphs—namely, that if G is a regular 2-connected bipartite graph of order p and valency ρ, where $\rho \geqslant \frac{1}{6}p$, then G is Hamiltonian.

In the case of regular graphs, Jackson [**38**] has improved Nash-Williams' result on the number of edge-disjoint Hamiltonian circuits (see after Corollary 4.3), by proving that if G is a regular graph of order p ($\geqslant 14$) and valency ρ, where $\rho \geqslant \frac{1}{2}(p-1)$, then G contains at least $[\frac{1}{6}(3\rho - p + 1)]$ edge-disjoint Hamiltonian circuits (see also Section 11).

Our next result is due to Woodall [**68**]; he has also obtained many interesting results concerning the "binding number" of a graph.

Theorem 4.12. *If, for every non-empty subset S of V, the number of vertices adjacent to some vertex in S is at least* $\frac{1}{3}(|S|+p+3)$, *then G is Hamiltonian.* $\parallel$

We conclude this section by mentioning that several of the above results can be modified to yield sufficient conditions for a graph to be traceable, by means of the technique described in the proof of Theorem 2.1. As an example, we present the analog of Chvátal's theorem (Theorem 4.7); this analog can be used to prove a result of Clapham (1974) and Camion (1975) that every self-complementary graph is traceable:

Theorem 4.13. *Let G be a graph with non-decreasing valency-sequence* $\rho_1, \rho_2, \ldots, \rho_p$. *If* $\rho_k \leqslant k-1 < \frac{1}{2}(p-1) \Rightarrow \rho_{p-k+1} \geqslant p-k$, *for each* k, *then G is traceable.* ||

5. Other Sufficient Conditions

In this section we consider three topics—the independence number of a graph, powers of graphs, and line graphs. References not given explicitly in this section may be found in [43]. We start with a theorem of Chvátal and Erdős (1972):

Theorem 5.1. *Let G be a graph with connectivity* κ *and independence number* α. *If* $\alpha \leqslant \kappa$, *then G is Hamiltonian.*

Proof. Suppose that G is a non-Hamiltonian graph with connectivity κ and independence number $\alpha \leqslant \kappa$; we shall derive a contradiction. Since G is κ-connected, G must contain a circuit of length at least κ. Let C be a circuit of maximum length in G, and let v be a vertex not belonging to C. Since G is κ-connected, there exist κ paths from v to C which are vertex-disjoint apart from v, and intersect C only at the terminal vertices $v_1, \ldots, v_\kappa$. (It suffices to add a new vertex w joined to all the vertices of C, and to use Menger's theorem to deduce the existence of κ paths from v to w.) If for each i, w_i is the successor of v_i in some fixed cyclic ordering of C, then no w_i is joined to v, since otherwise, we can obtain a circuit of greater length than C by replacing the edge v_iw_i by the path from v_i to v together with the edge vw_i. Since $\alpha \leqslant \kappa$, the set $\{v, w_1, \ldots, w_\kappa\}$ is not an independent set, and so there exists an edge w_sw_t. By deleting the edges v_sw_s and v_tw_t, and adding the edge w_sw_t together with the paths joining v to v_s and v to v_t (see Fig. 9), we obtain a circuit of greater length than C, thereby giving the required contradiction. ||

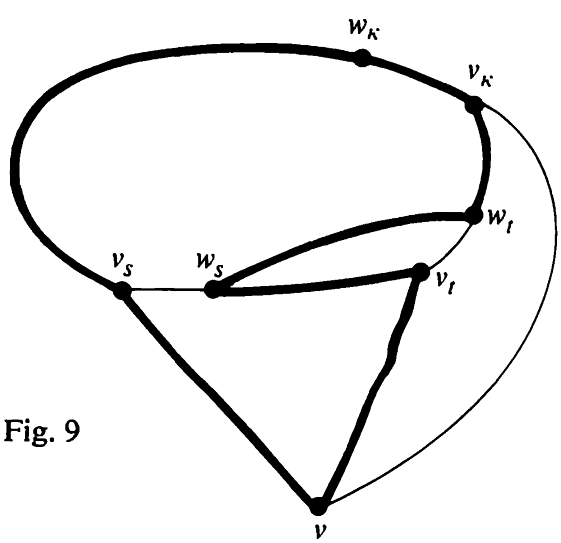


Fig. 9

Theorem 5.1 is best possible, in the sense that there exist non-Hamiltonian graphs for which $\alpha = \kappa+1$. Examples of such graphs are the complete bipartite graphs $K_{\kappa,\kappa+1}$, and the Petersen graph (with $\alpha = 4, \kappa = 3$). However, many of these non-Hamiltonian graphs G satisfy $t(G) < 1$. In this connection, the following results of Bilgáke are interesting (see **[9]**):

Theorem 5.2. *Let G be a κ-connected 1-tough graph of order p with independence number $\alpha \leqslant \kappa+1$.* Then

(*i*) *if $\kappa = 3$ and $p \geqslant 11$, then G is Hamiltonian*;

(*ii*) *if $\kappa = 4$, then G is Hamiltonian*;

(*iii*) *for each κ there exists a number $p_0(\kappa)$ such that if $p \geqslant p_0(\kappa)$, then G is Hamiltonian.* ||

From Theorems 3.1 and 3.3 we see that if $\kappa/\alpha \leqslant t(G) < 1$, then G is non-Hamiltonian, and by comparing this with the result of Theorem 5.1 we see that the only graphs which we have not classified as being Hamiltonian or not are those satisfying $\kappa/\alpha < 1 \leqslant t(G)$.

It is worth noting that Theorem 5.1 can be viewed as a generalization of Ore's theorem, as was observed by Bondy **[15]**, who proved the following result:

Theorem 5.3. *If $\rho(v)+\rho(w) \geqslant p$ for each pair of non-adjacent vertices v and w in a graph, then $\alpha \leqslant \kappa$.* ||

Bondy and Nash-Williams (1971) have also found a result which is stronger than Theorem 5.1 when the minimal valency $\rho_{\min}$ of G is not too small.

Theorem 5.4. *If G is a 2-connected graph with $\rho_{\min} \geqslant \max\{\alpha, \frac{1}{3}(p+2)\}$, then G is Hamiltonian.* ||

Powers of Graphs

The kth power of a graph G, denoted by G^k, is the graph whose vertices correspond to those of G, and where two distinct vertices are joined whenever the distance between them in G is at most k. There are several results relating Hamiltonian graphs to powers of graphs, including the following theorem of Sekanina (1960):

Theorem 5.5. *The cube of every connected graph is Hamiltonian.*

Remark. Using induction on p (the order of the graph G), one can prove the stronger result that G^3 is "Hamiltonian-connected"—that is, that between any two vertices of G^3 there is a Hamiltonian path; such a proof may be found in **[6**, Chapter 10**]**. The proof we shall outline here is an algorithmic proof due essentially to Rosenstiehl (1971); the construction given here yields a Hamiltonian circuit of G^3 in $O(p)$ steps.

Proof. We note first that the theorem can be rephrased as: the vertices of any connected graph G can be cyclically ordered so that the distance between any two consecutive vertices is at most 3. Note also that it is sufficient to restrict our attention to the case where G is a tree—the general result follows by regarding this tree as a spanning tree of G.

So let T be a tree of order p, and let T^* be the symmetric digraph obtained by replacing each edge of T by two opposite arcs. Let $\vec{P}_{2p-2}$ be an Eulerian directed path in T^*; such a path can be constructed by means of the following algorithm (an example is given at the end of the proof):

Step 1. Choose an arbitrary arc a_1, and let $\vec{P}_1 = a_1$.

Step 2. Suppose that $\vec{P}_k = a_1, a_2, \ldots, a_k$, and that there exists an arc a_{k+1} whose initial vertex is the end-vertex of a_k, and such that neither a_{k+1} nor its opposite occurs in $\vec{P}_k$; then we define $\vec{P}_{k+1} = a_1, a_2, \ldots, a_k, a_{k+1}$, and repeat Step 2 with $\vec{P}_{k+1}$ instead of $\vec{P}_k$. If no such arc a_{k+1} exists, go to Step 3.

Step 3. Suppose that $\vec{P}_k = a_1, a_2, \ldots, a_k$, and that there exists an arc a_{k+1} not belonging to $\vec{P}_k$ whose initial vertex is the end-vertex of a_k; then we define $\vec{P}_{k+1} = a_1, a_2, \ldots, a_k, a_{k+1}$, and repeat Step 2 with $\vec{P}_{k+1}$ instead of $\vec{P}_k$. If no such arc a_{k+1} exists, go to Step 4.

Step 4. STOP.

It is easy to show that when Step 4 is applied, the directed path obtained is Eulerian. Also, when Step 3 is applied, the resulting arc a_{k+1} is uniquely determined, since T is a tree. Moreover, if an arc appears in an odd position in the path, then its opposite must appear in an even place—in fact, if the part of the path between an arc and its opposite contains an arc, then it necessarily contains its opposite. Thus the sequence of $p-1$ "odd arcs" $a_1, a_3, a_5, \ldots, a_{2p-3}$ gives an ordering of the edges of T. (Note that this ordering has the property that two edges with consecutive labels are either adjacent, or incident with the end vertices of a third one; this proves essentially that the square of the line graph of T is Hamiltonian—a special case of Theorem 5.10(*ii*).)

From the sequence of arcs $a_1, a_3, \ldots, a_{2p-3}$ we can deduce the required ordering of the vertices, as follows: label the initial vertex of the arc a_1 with label 0, and the terminal vertex of a_1 with label 1; then assign each label k to the initial or terminal vertex of the arc a_{2k-1} according as this arc comes after, or before, its opposite in $\vec{P}_{2p-2}$. This labeling can be shown to have the required properties, and the proof is complete. ‖

As an example of the above procedure, let T be the tree shown in Fig. 10; the direction of the arcs a, b, c, d, e, f in T^* is taken to be from left to right. Then the algorithm above gives us, for example,

$$\vec{P}_{2p-2} = \vec{a}, \vec{b}, \vec{c}, \vec{d}, \overleftarrow{d}, \overleftarrow{c}, \vec{e}, \vec{f}, \overleftarrow{f}, \overleftarrow{e}, \overleftarrow{b}, \overleftarrow{a}.$$

The sequence $a_1, a_3, \ldots, a_{2p-3}$ is $\vec{a}, \vec{c}, \overleftarrow{d}, \vec{e}, \overleftarrow{f}, \overleftarrow{b}$, and we get the ordering of the vertices indicated in Fig. 10. (Note that, for example, 3 appears as the initial vertex of $\overleftarrow{d}$, since $\overleftarrow{d}$ appears after its opposite $\vec{d}$ in $\vec{P}_{2p-2}$.)

It follows from the example of Fig. 10 that the square of a connected graph graph is not necessarily Hamiltonian. In fact, if T is a tree, then Neumann

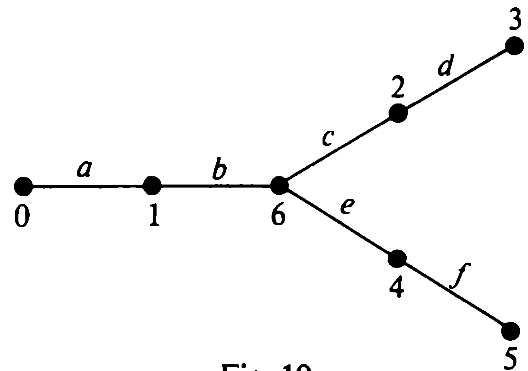


Fig. 10

(1964), and Harary and Schwenk (1971), proved that T^2 is Hamiltonian if and only if T does not contain the tree of Fig. 10 as a subgraph. The characterization of those graphs whose square is Hamiltonian is an unsolved problem*, but Fleischner (1974) has obtained the following deep result, thereby answering a conjecture of Plummer and Nash-Williams:

Theorem 5.6 (Fleischner's Theorem). *The square of every* 2*-connected graph is Hamiltonian.* ‖

The most difficult and important part of the proof of Fleischner's theorem is the proof of the fact that every connected bridgeless graph contains an "*EPS*-subgraph"—that is, a connected spanning subgraph S which is the edge-disjoint union of a (not necessarily connected) graph E, all of whose vertices have even valency, with a (possibly empty) forest P each of whose components is a path. The proof of this result, and its use in the proof of Fleischner's theorem, are too complicated to be given here. As we remarked in Section 3, Fleischner's theorem would also follow from Chvátal's conjecture that every graph G with toughness $t(G) \geqslant 2$ is Hamiltonian.

Nebeský has observed that Fleischner's theorem implies the following result:

* See P. Underground, *Discrete Math.* **21** (1978), 323.

Theorem 5.7. *If G is any graph, then either G^2 or $(\bar{G})^2$ is Hamiltonian.*

Proof. If G is 2-connected, then G^2 is Hamiltonian, by Fleischner's theorem. If G is connected, but not 2-connected, then G contains a cut-vertex v. If $\rho(v) = p-1$, where p is the order of G, then G^2 is a complete graph, and the result is clear. If $\rho(v) < p-1$, then $(\bar{G}-v)^2$ is a complete graph, and so $(\bar{G})^2$ is Hamiltonian. ‖

Fleischner and Hobbs **[27]** have shown that if G^2 is Hamiltonian, then G must have a connected spanning subgraph which resembles an *EPS*-subgraph, except that the components of P are those which do not contain the tree of Fig. 10 as a subgraph. They also obtained new classes of graphs whose square is Hamiltonian (see, for example, **[36]**).

They have also reformulated some of their results in terms of total graphs (see Chapter 1), whose relevance here derives from the fact that if $T(G)$ is the total graph of a graph G, then $T(G)$ is the square of the subdivision graph $S(G)$ obtained by inserting a vertex into each edge of G—for example, the tree in Fig. 10 is $S(K_{1,3})$. Note that Fleischner's theorem implies that if G is 2-connected, then $T(G)$ is Hamiltonian. It can also be shown that if G is 2-edge-connected or planar, then $T(G)$ is Hamiltonian. These results follow from the following characterization of Hamiltonian total graphs obtained by Fleischner and Hobbs **[28]**.

Theorem 5.8. *$T(G)$ is Hamiltonian if and only if G contains an EPS-subgraph.* ‖

Hamiltonian Line Graphs

Although line graphs are studied in detail in Chapter 10, we include here a few results which pertain to Hamiltonian graphs (for full references, see Chapter 10, or **[43]**). One of these is the following theorem of Harary and Nash-Williams (1965):

Theorem 5.9. *The line graph $L(G)$ is Hamiltonian if and only if either G is isomorphic to $K_{1,s}$ for some $s \geqslant 3$, or G contains a closed path C with the property that every edge of G is incident to at least one vertex of C.* ‖

Using Theorem 5.9, we can easily deduce that if G is either Hamiltonian or Eulerian, then $L(G)$ is Hamiltonian. One can also prove the following results, due to Nebeský (1973) and Bermond and Rosenstiehl (1973):

Theorem 5.10. *Let G be a connected graph of order p. Then*

(*i*) *if $p \geqslant 5$, then at least one of the graphs $L(G)$ and $L(\bar{G})$ is Hamiltonian;*

(*ii*) *if* $p \geqslant 4$, *then* $L^2(G)$ *is Hamiltonian*;

(*iii*) *if* $p \geqslant 3$, *then* $L(G^2)$ *is Hamiltonian*. ||

The proof of part (*ii*) is obtained by a slight refinement of the algorithm used in the proof of Theorem 5.5.

We conclude this section with a result of Chartrand (1968) on the "iterated line graph", defined for each $k > 1$ by $L^k(G) = L(L^{k-1}(G))$:

Theorem 5.11. *If G is a connected graph of order p (other than a path), then* $L^k(G)$ *is Hamiltonian for all* $k \geqslant p-3$. ||

6. Hamiltonian Planar Graphs

The interest in Hamiltonian planar graphs arises partly from a result of Whitney (1931), that in order to prove the four-color theorem it is sufficient to consider only Hamiltonian planar graphs, and partly from a related conjecture of Tait (1880), that every trivalent 3-connected planar graph is Hamiltonian. (Fuller references for the results in this section may be found in Grünbaum [34].) If Tait's conjecture had been proved true, we could have obtained a very simple proof of the four-color theorem, but a counter-example of order 46 was found by Tutte (1946) (see Fig. 11). The importance of 3-con-

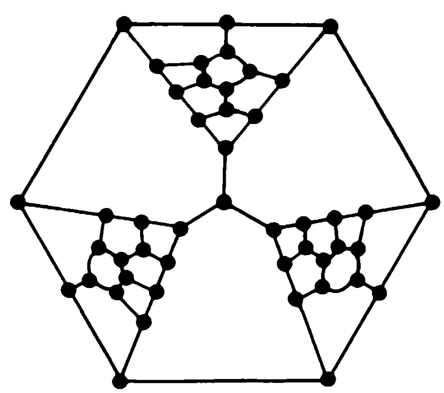


Fig. 11

nected planar graphs is due primarily to their relationship with polyhedra. A graph G is a ***d*-polytopal graph** if there exists a d-dimensional convex polytope P whose vertices and edges are in one-to-one correspondence with the vertices and edges of G. A 3-polytopal graph is called a **polyhedral graph**, and such graphs have been characterized by Steinitz (1922):

Theorem 6.1. *A graph G is polyhedral if and only if it is planar and 3-connected*. ||

It follows from Tutte's counter-example (Fig. 11) that there exist trivalent polyhedral graphs which are not Hamiltonian; the smallest example of a polyhedral non-Hamiltonian graph is the "Herschel graph" shown in Fig. 12. Note also that the Petersen graph is a trivalent 3-connected non-Hamiltonian graph. A method for generating polyhedral non-Hamiltonian graphs has

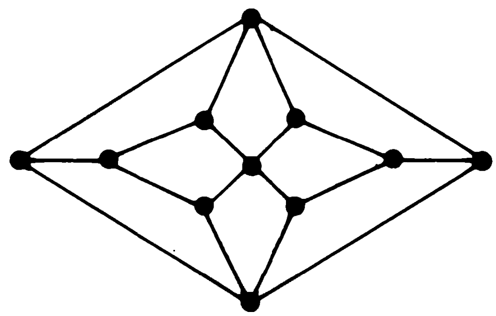


Fig. 12

been described by Grinberg (1968), who has also given the following necessary condition for a planar graph to be Hamiltonian:

Theorem 6.2. *Let G be a Hamiltonian planar graph, and let C be a Hamiltonian circuit in G. Let f_k denote the number of regions bounded by k edges in the interior of C, and let g_k denote the number of regions bounded by k edges in the exterior of C. Then*

$$\sum_k (k-2)(f_k - g_k) = 0.$$

Proof. If m denotes the number of edges of G which lie in the interior of C, then the total number of regions in the interior of C (that is, $\sum_k f_k$) is $m+1$, since C is a Hamiltonian circuit. But each such edge lies on the boundary of two regions in the interior of C, and each edge of C lies on the boundary of one region in the interior of C, so that $\sum_k kf_k = 2m+p$, where p is the order of G. It follows that

$$\sum_k (k-2) f_k = p-2,$$

and similarly that

$$\sum_k (k-2) g_k = p-2.$$

The result follows immediately. ||

As an example of the use of this theorem, we shall prove that the planar graph of Fig. 13 is non-Hamiltonian. In this graph, every region is either a pentagon or an octagon, except for one which is a square, and so, if the graph were Hamiltonian, we should have the equation

$$3(f_5-g_5)+6(f_8-g_8)+2(f_4-g_4) = 0.$$

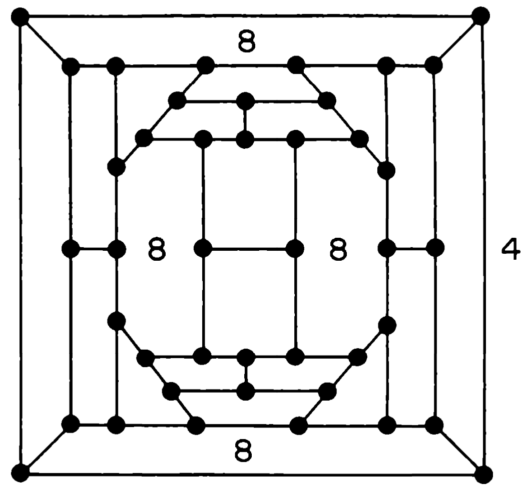


Fig. 13

But this is impossible, as can be seen by reducing modulo 3; the graph of Fig. 13 is therefore non-Hamiltonian.

Related to the above ideas is that of a cyclically edge-connected graph. A graph G is called **cyclically k-edge connected** if by deleting fewer than k edges, we cannot disconnect G into components each of which contains a circuit; for example, the graphs in Figs 11 and 13 are cyclically 3-edge connected, and cyclically 4-edge connected, respectively. The interest in such graphs arises from the fact that if one could prove that every trivalent 3-connected planar graph which is cyclically 5-connected is also Hamiltonian, then one could deduce the four-color theorem. However, this is not the case—an example of a trivalent 3-connected planar graph which is both cyclically 5-connected and non-Hamiltonian is given in Fig. 14; it is due to Grinberg.

Whitney (1931) proved that every plane triangulation without separating

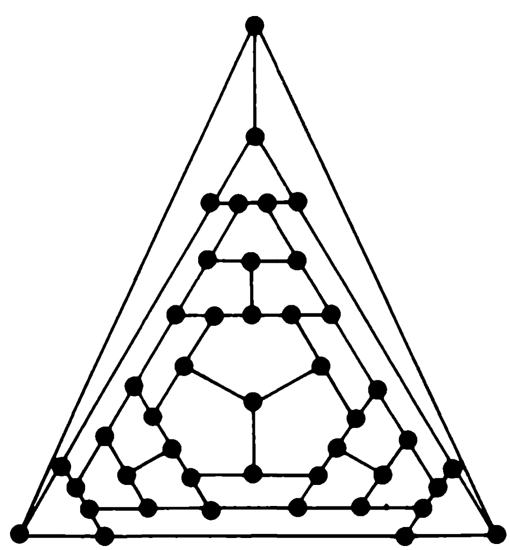


Fig. 14

triangles (triangles which do not bound a region) is Hamiltonian. Tutte (1956) generalized this to give the following important theorem, whose proof is much too complicated to be given here; a comprehensive proof is given in **[64]**, and an algorithm in $O(p^3)$ steps to find a Hamiltonian circuit may be found in **[32]**:

Theorem 6.3. *Every* 4-*connected planar graph is Hamiltonian.* ||

The condition of planarity cannot be omitted here, as shown by the graph $K_{4,5}$. Meredith (see **[17**, p. 239]) has exhibited a regular 4-valent, 4-connected non-Hamiltonian graph, thereby disproving a conjecture of Nash-Williams. The depth of Tutte's theorem is shown by the fact that Malkevitch (1971) and others have constructed 4-connected planar graphs which are not pancyclic (see Section 8), and which contain no pair of disjoint Hamiltonian circuits (see Grünbaum and Malkevitch **[35]**, Martin **[46]** and Owens **[54]**).

Various other results of a similar nature have been obtained. In the following theorem we summarize some of these results, referring the reader to the surveys of Grünbaum **[33]**, **[34]** and Owens **[54]** for references and further details:

Theorem 6.4. (*i*) *There exist d-polytopal non-Hamiltonian graphs, for every* $d \geqslant 3$;

(*ii*) *there exist regular* 4-*valent and* 5-*valent graphs which are* 3-*connected, cyclically* 6-*edge connected, planar, and non-Hamiltonian*;

(*iii*) *there exist regular* 3-*valent*, 4-*valent and* 5-*valent graphs which are* 3-*connected, planar, and non-traceable*;

(*iv*) *there exist constants* $\alpha < 1$ *and* $c > 0$ *with the property that there is a* 3-*connected planar graph of order p the length of whose longest path is less than* cp^{α}. ||

For bipartite planar graphs, Barnette **[3]** has formulated the following conjecture:

Conjecture. *Every regular* 3-*valent* 3-*connected bipartite planar graph is Hamiltonian.*

Partial results have been obtained by Goodey **[31]** and Peterson **[55]**. Horton has shown that this conjecture becomes false if the condition of planarity is omitted, thereby settling a conjecture of Tutte [see **17**, p. 240].

We conclude this section by mentioning briefly some results involving Hamiltonian graphs on surfaces of positive genus g. The first of these is due to Duke (1972):

Theorem 6.5. *For each $g \geqslant 1$, there exists an integer $c(g)$ such that every $c(g)$-connected graph of genus g is Hamiltonian. Furthermore, $c(g)$ satisfies the inequalities*

$$[\tfrac{1}{2}(5+\sqrt{1+16g})] \leqslant c(g) \leqslant \{3+\sqrt{3+6g}\}. \quad \|$$

A special case of this result (for $g = 1$) had previously been proved by Altshuler (1972), and Bloom and Schmeichel **[11]** have also shown that Altshuler's result follows from Chvátal's conjecture in Section 3 that every graph of toughness $t \geqslant 2$ is Hamiltonian. In fact, Bloom and Schmeichel's observation is the special case $g = 1$, $k = 6$ of the following more general theorem **[11]**:

Theorem 6.6. *If G is a k-connected graph of genus g, then*

$$c(G-S) \leqslant \frac{2}{k-2}(|S|-2+2g),$$

for all subsets $S \subseteq V(G)$ such that $|S| > k$. $\|$

7. Hamiltonian Digraphs

In view of Theorem 2.1, one might expect that most of the results in this chapter extend easily to digraphs. In fact, this is not the case—some results have no analogs for digraphs, while others can be extended but the proofs are more complicated and the methods are different. In this section we shall denote by $\rho_{\text{out}}(v)$ the out-valency of a vertex v in a digraph D (the number of arcs of D of the form vw), and by $\rho_{\text{in}}(v)$ the in-valency of v (the number of arcs of D of the form wv). The valency $\rho(v)$ of v is defined to be the sum of $\rho_{\text{out}}(v)$ and $\rho_{\text{in}}(v)$.

We first give the extension to digraphs of Ore's theorem (Corollary 4.2), which was obtained by Meyniel **[47]**; simpler proofs have been given by Overbeck-Larisch **[53]**, and Bondy and Thomassen **[18]**. The proof presented here is a constructive version of Bondy and Thomassen's proof, following an idea of Minoux **[48]**; it shows that a Hamiltonian circuit can be constructed in $O(p^4)$ steps.

Theorem 7.1 (Meyniel's Theorem). *Let D be a strongly-connected digraph of order p. If $\rho(v)+\rho(w) \geqslant 2p-1$ for each pair of non-adjacent vertices v and w in D, then D is Hamiltonian.*

Proof. We shall need the following notation: if S is any subset of V, the vertex-set of D, then $\rho_S(v)$ will denote the number of arcs joining v with vertices in S (in either direction); an *S-path* is a directed path of length two or more whose ends belong to S, but whose vertices are otherwise disjoint from S. A directed path $v_0 \to v_1 \to \ldots \to v_k$ is called *quasi-maximum* if there exists no other vertex v such that $v_i v$ and vv_{i+1} are both arcs of D, for some i; similarly, a directed circuit $v_0 \to v_1 \to \ldots \to v_k \to v_0$ is called quasi-maximum if there exists no other vertex v such that $v_i v$ and vv_{i+1} are both arcs of D, for some i, or such that $v_k v$ and vv_0 are both arcs of D.

(1) We first prove that if $P = v_0 \to v_1 \to \ldots \to v_k$ is a quasi-maximum directed path in D, then for every vertex v not included in P,

$$\rho_{V(P)}(v) \leqslant |V(P)|+1.$$

In order to see this, we let $A = \{v_i : 1 \leqslant i \leqslant k-1$, and $v_i v$ is an arc of $D\}$ and $B = \{v_i : 1 \leqslant i \leqslant k-1$, and vv_{i+1} is an arc of $D\}$. Then $A \cap B = \varnothing$, by hypothesis, and $v_k \notin A \cup B$, so that $|A|+|B| \leqslant |V(P)|-1$. It follows that $\rho_{V(P)}(v) \leqslant |A|+|B|+2 \leqslant |V(P)|+1$, as required.

(2) In order to construct a Hamiltonian directed circuit in D, we start by constructing a quasi-maximum directed circuit C. Note that the construction of such a directed circuit involves $O(p^3)$ steps, since in $O(p^2)$ steps one can check whether a circuit is quasi-maximum, or construct a directed circuit containing one more vertex, and there are p vertices to consider. If C is Hamiltonian, the construction is complete. If not, we can construct a longer circuit C' in $O(p^3)$ steps, and then a quasi-maximum circuit C'' longer than C, also in $O(p^3)$ steps; the details of these constructions are given in (3)–(6) below. Repeating the construction with C'', we obtain a Hamiltonian directed circuit in $O(p^4)$ steps.

(3) We now show that if C is a quasi-maximum directed circuit which is not Hamiltonian, then D contains an S-path, where S is the set of vertices of C. If no such path exists, then since D is strongly connected, D must contain a directed circuit C' having exactly one vertex z in common with C. Let S' be the set of vertices of C'. Since D has no S-path, no vertex in $S'-\{z\}$ can be joined to a vertex in $S-\{z\}$. Thus, for any $v \in S'-\{z\}$ and $w \in S-\{z\}$, we have $\rho_S(v) \leqslant 2$ and $\rho_S(w) \leqslant 2|S|-2$. Also, since D has no S-path, there cannot exist paths of the form $w \to u \to v$ or $v \to w$, for any $u \in V-S-\{v\}$, and so $\rho_{V-S}(v)+\rho_{V-S}(w) \leqslant 2(p-|S|-1)$. It follows that $\rho(v)+\rho(w) \leqslant 2p-2$, contradicting the hypothesis.

Note that the family of S-paths starting at a vertex v of C can be constructed in $O(p^2)$ steps.

(4) By (3), there exists an S-path in D. We shall take this path to be $P = x \to \ldots \to y$, where x and y lie in C, and we shall choose P so that the directed path $C(x, y)$ included in C with initial and terminal vertices x and y is as short as possible. (This choice can be made in $O(p^2)$ steps.) Let $S_1 = V(C(x, y))-\{x\}-\{y\}$, and $S_2 = S-S_1$, and let Q be a quasi-maximum

directed path joining y to x included in the subgraph generated by S and containing all of the vertices of S_2 (by (2), such a directed path can be chosen in $O(p^3)$ steps). We shall show that Q contains every vertex of S, so that $C'' = Q \cup P$ is a directed circuit which is longer than C. To do this, let $S_3 = V(Q)$, so that $S_2 \subseteq S_3 \subseteq S$; we shall assume that $S_3 \neq S$, and derive a contradiction.

(5) If $S_3 \neq S$, we have $S_2 \neq S$, and hence $S_1 \neq \varnothing$. By our choice of P, no vertex in $V(P)-\{x\}-\{y\}$ can be adjacent to any vertex in S_1, and so if $v \in V(P)-\{x\}-\{y\}$, we have $\rho_{S_1}(v) = 0$. By applying the result of (1) to the directed path included in C with initial and terminal vertices y and x, we get $\rho_{S_2}(v) \leqslant |S_2|+1$. Similarly, by applying the result of (1) to Q and the subgraph induced by S, we get $\rho_{S_3}(w) \leqslant |S_3|+1$, for any $w \in S-S_3$, and hence

$$\rho_S(w) \leqslant |S_3|+1+2(|S|-|S_3|-1) = 2|S|-|S_3|-1.$$

Finally, by our choice of P, there cannot exist paths $v \rightarrow u \rightarrow w$ or $w \rightarrow u \rightarrow v$ with $u \in V-S-\{v\}$, and so

$$\rho_{V-S}(v)+\rho_{V-S}(w) \leqslant 2(p-|S|-1).$$

Combining all these inequalities gives

$$\rho(v)+\rho(w) \leqslant |S_2|+1+2|S|-|S_3|-1+2(p-|S|-1) \leqslant 2p-2,$$

which contradicts the hypothesis. This contradiction proves the theorem. ||

Using Theorem 7.1 we can easily deduce a theorem of Ghouila-Houri (1960), which is the digraph analog of Dirac's theorem (Corollary 4.3); for other corollaries, see **[6]**, **[47]**, or Woodall (1972):

Corollary 7.2. *Let D be a strongly-connected digraph of order p. If $\rho(v) \geqslant p$ for each vertex in D, then D is Hamiltonian.* ||

The next corollary is due to Camion (1959); for a stronger result, with a direct proof, see Theorem 3.3 of Chapter 7:

Corollary 7.3. *Every strongly connected tournament is Hamiltonian.*

Proof. Since every two vertices are adjacent in a tournament, there is no pair of non-adjacent vertices, and so the result follows trivially from Theorem 7.1. ||

The requirement in Theorem 7.1 and Corollaries 7.2 and 7.3 that D be strongly connected is necessary, as can be seen by considering a digraph consisting of two complete symmetric digraphs joined by arcs all in the same direction. If we do remove this requirement, then we get the following weaker result:

Corollary 7.4. *Let D be a digraph of order p. If $\rho(v)+\rho(w) \geqslant 2p-3$ for each pair of non-adjacent vertices v and w in D, then D is traceable.*

Proof. We use the trick indicated in the proof of Theorem 2.1. Let $\tilde{D}$ be the digraph obtained from D by adding a new vertex and joining it to every vertex of D by two opposite arcs. Then $\tilde{D}$ is a strongly connected digraph which satisfies the conditions of Theorem 7.1 (with p replaced by $p+1$). It follows that $\tilde{D}$ is Hamiltonian, and hence that D is traceable. ‖

As a corollary of this last result, we can also obtain Corollary 3.2 of Chapter 7, that every tournament is traceable.

One can also try to obtain digraph analogs of various other theorems for Hamiltonian graphs—for example, Chvátal's theorem (Theorem 4.7). In particular, we can ask whether every strongly connected digraph whose non-decreasing valency-sequence $\rho_1, \ldots, \rho_p$ satisfies the following conditions is Hamiltonian:

(*i*) $\rho_k \leqslant 2k < p \Rightarrow \rho_{p-k} \geqslant 2(p-k)$, for each k.

Similarly, one can ask whether every strongly connected digraph whose non-decreasing out-valency and in-valency sequences $\rho_1^+, \ldots, \rho_p^+$ and $\rho_1^-, \ldots, \rho_p^-$ satisfy the following conditions is Hamiltonian:

(*ii*) $\rho_k^+ \leqslant k < \frac{1}{2}p \Rightarrow \rho_{p-k}^+ \geqslant p-k$ and $\rho_k^- \leqslant k < \frac{1}{2}p \Rightarrow \rho_{p-k}^- \geqslant p-k$.

In fact, both of these are false in general, as is shown by the following simple example: let D be a complete symmetric digraph of order $p-2$ containing a vertex u, and let $\tilde{D}$ be the digraph obtained by adding two new vertices v and w dominated by u, and dominating every other vertex of D. Then $\tilde{D}$ is a strongly connected digraph whose valency-sequences satisfy (*i*) and (*ii*), but which is not Hamiltonian.

A third possible analogue of Chvátal's theorem, which survives the above example, is the following conjecture due to Nash-Williams **[52]**:

Conjecture. *If D is a strongly connected digraph whose non-decreasing out-valency and in-valency sequences $\rho_1^+, \ldots, \rho_p^+$ and $\rho_1^-, \ldots, \rho_p^-$ satisfy the conditions*

$$\rho_k^+ \leqslant k < \tfrac{1}{2}p \Rightarrow \rho_{p-k}^- \geqslant p-k \qquad \text{and} \qquad \rho_k^- \leqslant k < \tfrac{1}{2}p \Rightarrow \rho_{p-k}^- \geqslant p-k$$

for each k, then D is Hamiltonian.

By using Corollary 7.2, Lewin **[44]** has shown that every strongly connected digraph of order p with at least $(p-1)(p-2)+3$ arcs is Hamiltonian, thereby providing a partial analog of Theorem 4.9. Furthermore, it can be shown that the only non-Hamiltonian digraphs of order p with $(p-1)(p-2)+2$ arcs are the symmetric digraphs corresponding to the extremal graphs $G(1, p)$ and $G(2, 5)$ of Theorem 4.9, and the above example. On the other hand, the example of Fig. 15 (due to Las Vergnas), representing a strongly 2-connected digraph whose independence number is 2, shows the difficulty of extending the theorem of Chvátal and Erdős (Theorem 5.1). Similarly, Fouquet [29] has exhibited, for every k, a strongly k-connected digraph D with the property that D^k is non-Hamiltonian, thereby showing that there is no analog of Theorem 5.5.

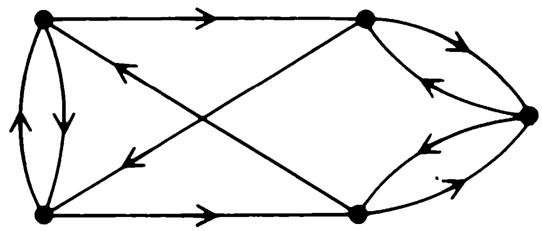


Fig. 15

It seems that the study of Hamiltonian digraphs is generally much more difficult than the corresponding study for graphs, but we conclude this section with a result of Kasteleyn (1963) on line digraphs, where the digraph version is the simpler one (see also Chapter 10, Section 10):

Theorem 7.5. *$L(D)$ is Hamiltonian if and only if D is Eulerian.* □

8. Pancyclic and Panconnected Graphs

A graph G of order p is said to be **pancyclic** if G contains a circuit of length l, for each l satisfying $3 \leqslant l \leqslant p$; in particular, every pancyclic graph is Hamiltonian. The concept of a pancyclic graph was introduced by Bondy, who has written two expository papers on the subject [13], [14] (see also [43] for references), and who has made the following "meta-conjecture": every condition which implies that a graph is Hamiltonian also implies that it is also pancyclic, with the possible exception of a simple family of exceptional graphs. Although this meta-conjecture is sometimes false, it turns out to be accurate in a surprisingly large number of cases.

We start with a result of Hakimi and Schmeichel (1974) which extends Chvátal's theorem (Theorem 4.7); the argument for the case when p is odd is due to Bondy (see [21, p. 92]):

Theorem 8.1. *Let G be a graph with non-decreasing valency-sequence $\rho_1, \rho_2, \ldots, \rho_p$. If $\rho_k \leqslant k < \frac{1}{2}p \Rightarrow \rho_{p-k} \geqslant p-k$, for each k, then G is either pancyclic or bipartite.*

Sketch of Proof. Suppose first that p is odd. Then $\rho_{\frac{1}{2}(p+1)} > \frac{1}{2}p$, since either $\rho_{\frac{1}{2}(p-1)} > \frac{1}{2}p$, in which case the result is obvious, or $\rho_{\frac{1}{2}(p-1)} \leqslant \frac{1}{2}(p-1)$, in which case (by the hypothesis) $\rho_{\frac{1}{2}(p+1)} \geqslant \frac{1}{2}(p+1) > \frac{1}{2}p$.

Now, by Chvátal's theorem, G is Hamiltonian. Let $v_1, v_2, \ldots, v_p$ be a Hamiltonian circuit, and suppose that G is not pancyclic and thus does not contain a circuit of length l, for some l satisfying $3 \leqslant l < p$. It follows that we cannot have simultaneously in G the pairs of edges

$$v_jv_k \quad \text{and} \quad v_{j+1}v_{k-l+3}, \quad \text{for} \quad j+l-1 \leqslant k \leqslant j-1,$$

and

$$v_jv_k \quad \text{and} \quad v_{j+1}v_{k-l+1}, \quad \text{for} \quad j+2 \leqslant k \leqslant j+l-2$$

(where all suffixes are taken modulo p—see Fig. 16), and so we have $\rho(v_j)+\rho(v_{j+1}) \leqslant p$.

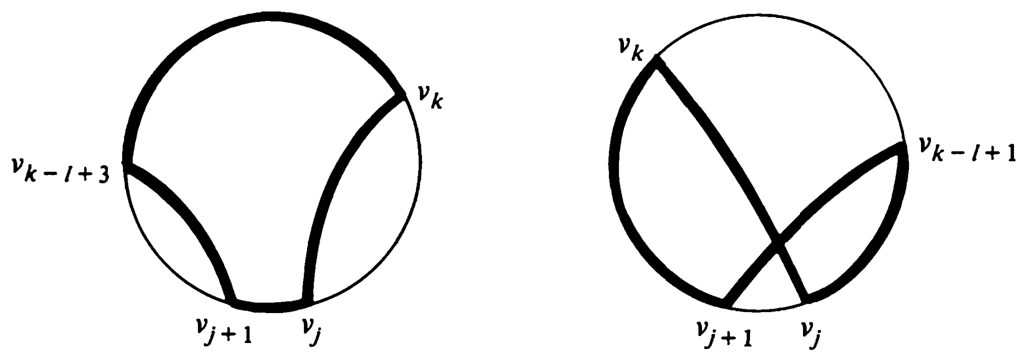


Fig. 16

But since $\rho_{\frac{1}{2}(p+1)} > \frac{1}{2}p$, more than half of the vertices of G have valency greater than $\frac{1}{2}p$, and so two such vertices (v_j and v_{j+1}, say) must be adjacent in the Hamiltonian circuit and $\rho(v_j)+\rho(v_{j+1}) > p$, which is the required contradiction.

If p is even and $\rho_{\frac{1}{2}p} \neq \frac{1}{2}p$, then the proof is the same as the one just given, but if $\rho_{\frac{1}{2}p} = \frac{1}{2}p$, then the proof is much more complicated, and will be omitted. ‖

It is interesting to note that although Chvátal's theorem generalizes, the result of Corollary 4.5 on the closure of a graph does not. In particular, Thomassen has exhibited graphs G whose closure cl(G) is complete, but which are neither pancyclic nor bipartite. However, Ore's theorem (Corollary 4.2) does generalize; the following result was proved by Bondy (1971) and can be deduced from Theorem 8.1:

Corollary 8.2. *Let G be a graph of order p ($\geqslant 3$). If $\rho(v)+\rho(w) \geqslant p$ for each pair of non-adjacent vertices v and w in G, then G is either pancyclic or the graph $K_{\frac{1}{2}p,\frac{1}{2}p}$.* ‖

There is also a generalization of the theorem of Chvátal and Erdős (Theorem 5.1), proved by Erdős in 1974:

Theorem 8.3. *Let G be a graph with connectivity κ and independence number α, and suppose that $\alpha \leqslant \kappa$. Then there exists a number c such that if G has at least $c\kappa^4$ vertices, then G is pancyclic.* ‖

In the case $\kappa = 2$, Bondy has shown that if $\alpha \leqslant 2$, then either G is pancyclic, or $G \cong C_4$ or C_5. More generally, he has shown that there exist two families of non-pancyclic graphs with $\alpha = \kappa$—namely, $K_{\kappa,\kappa}$, and the graph G_κ consisting of a circuit of length $2\kappa+2$ together with the edges $v_i v_{i+2j}$ for $1 \leqslant i \leqslant 2\kappa+2$, $2 \leqslant j \leqslant \kappa-1$.

Several of the results on powers of graphs (Section 5) also have pancyclic analogs. We summarize these in the following theorem (see **[13]** and **[26]**); part (*iii*) follows from part (*ii*), with the aid of Theorem 5.6:

Theorem 8.4. *Let G be a connected graph. Then*

(*i*) *G^3 is pancyclic;*

(*ii*) *if G^2 is Hamiltonian, then G^2 is pancyclic;*

(*iii*) *if G is 2-connected, then G^2 is pancyclic.* ‖

A corresponding concept also exists for digraphs. A digraph D of order p is said to be **pancyclic** if D contains a directed circuit of length l, for each l satisfying $3 \leqslant l \leqslant p$; in particular, every pancyclic digraph is Hamiltonian. Moon has shown that every strongly connected tournament is pancyclic, and this result is proved in Chapter 7 (Theorem 3.3). The following theorem was obtained by Thomassen **[62]**:

Theorem 8.5. *Let D be a strongly connected digraph of order p ($\geqslant$3). If $\rho(v)+\rho(w) \geqslant 2p$ for each pair of non-adjacent vertices v and w in D, then D is either pancyclic or the symmetric digraph obtained by replacing each edge of $K_{\frac{1}{2}p,\frac{1}{2}p}$ by two opposite arcs.* ‖

This theorem does not directly generalize Meyniel's theorem (Theorem 7.1) since there exist non-pancyclic digraphs satisfying the hypotheses of Theorem 7.1. However, it does contain as special cases Moon's tournament results, and earlier results of Overbeck-Larish, Häggkvist and Thomassen.

We now turn our attention to panconnected graphs. Following Ore, we define a graph G to be **Hamiltonian-connected** if each pair of vertices of G can be connected by a Hamiltonian path. A graph G of order p is **panconnected** if each pair v, w of vertices of G can be connected by a path of length l, for each l satisfying $d(v, w) \leqslant l \leqslant p-1$ (where $d(v, w)$ is the distance between v

and w). Note that every panconnected graph is Hamiltonian-connected and pancyclic. Faudree and Schelp (1976) have conjectured that if G is a Hamiltonian-connected graph of order p, then each pair of vertices of G can be connected by a path of length l, for each l satisfying $l \geqslant \frac{1}{2}p$*. Some properties of these graphs are summarized in the following theorem (see **[26]**); part (*i*) is due to Alavi and Williamson (1975) and part (*iii*) is due to Chartrand, Hobbs, Jung, Kapoor and Nash-Williams (1974):

Theorem 8.6. *Let G be a connected graph. Then*

(*i*) *G^3 is panconnected;*

(*ii*) *if G^2 is Hamiltonian-connected, then G^2 is panconnected;*

(*iii*) *if G is 2-connected, then G^2 is panconnected.* ‖

In order to obtain analogs for Hamiltonian-connected graphs of our earlier theorems on Hamiltonian graphs, we usually need to take a slightly stronger hypothesis. In many cases the proofs are very similar to the original ones, or the new results can easily be deduced from the original results. As an example, we give the following analog of Chvátal's theorem (Theorem 4.7):

Theorem 8.7. *Let G be a graph with non-decreasing valency sequence $\rho_1, \rho_2, \ldots, \rho_p$. If $\rho_k \leqslant k+1 < \frac{1}{2}(p+1) \Rightarrow \rho_{p-k-1} \geqslant p-k$, for each k, then G is Hamiltonian-connected.*

Proof. Let v and w be any pair of vertices of G, and let G' be the graph obtained from G by adding a new vertex z, joining it to v and w, and deleting the edge vw (if such an edge exists). Then there is a Hamiltonian path connecting v and w in G if and only if there is a Hamiltonian circuit in G'. Since G' satisfies the hypotheses of Chvátal's theorem (with p replaced by $p+1$), G' is Hamiltonian, and hence G is Hamiltonian-connected. ‖

9. Strongly Hamiltonian Graphs

A great number of generalizations of the ideas of the previous section have been investigated. Most of these are included in the following definition, proposed by Skupien and Wojda in 1974 (see the survey of Wojda **[65]**). A graph G is said to be **strongly *t*-edge Hamiltonian** if, for every system S of vertex-disjoint paths of total length t of the complete graph with vertex-set $V(G)$, there exists in the graph $G' = (V(G), E(G) \cup S)$ a Hamiltonian circuit containing S. In particular, a strongly 1-edge Hamiltonian graph is simply a Hamiltonian-connected graph. If all the edges of S belong to $E(G)$, then G is said to be t-edge Hamiltonian, a concept introduced by Pósa in 1963.

* This conjecture has been disproved by Thomassen.

Furthermore, G is said to be **strongly (*s*, *t*)-Hamiltonian** if the deletion of any s' vertices of G (where $0 \leqslant s' \leqslant s$) results in a strongly t-edge Hamiltonian graph; a strongly $(s, 0)$-Hamiltonian graph is called an **s-Hamiltonian graph**.

Note that these definitions are not equivalent. Although every Hamiltonian-connected graph is 1-edge Hamiltonian, the converse is not true, as may be seen by considering the circuit graph C_p. There also exist Hamiltonian-connected graphs which are not 1-Hamiltonian, and 1-Hamiltonian graphs which are not 1-edge Hamiltonian (see Theorem 10.3). Most of the results in Section 4 can be extended to give sufficient conditions for a graph to be strongly (s, t)-Hamiltonian. For example, in order to generalize Theorem 4.4, it is sufficient to consider the "$p+s+t$-closure" of G, defined to be the smallest graph H such that (*i*) G is a spanning subgraph of H, and (*ii*) $\rho_H(v)+\rho_H(w) < p+s+t$ for every pair of non-adjacent vertices v and w in H (see **[16]**). In a similar way, Chvátal's theorem (Theorem 4.7) becomes the following:

Theorem 9.1. *Let G be a graph with non-decreasing valency-sequence $\rho_1 \leqslant \rho_2 \leqslant \ldots \leqslant \rho_p$. If $\rho_k \leqslant k+s+t \leqslant \frac{1}{2}(p+s+t) \Rightarrow \rho_{p-s-t-k} \geqslant p-k$, for each k, then G is strongly (s, t)-Hamiltonian.* ‖

Many other results have been obtained. For example, Benhocine **[5]** has obtained the following theorem on powers of graphs:

Theorem 9.2. *If G is a connected graph, and k is a positive integer, then G^k is strongly (s, t)-Hamiltonian for all s and t satisfying $t > 0$ and $s+t \leqslant k-2$.* ‖

However, some of these generalizations remain to be proved. For example, the following conjecture seems reasonable, and reduces to Chvátal and Erdős's theorem (Theorem 5.1) when $s = t = 0$:

Conjecture A. *Let s, t and κ be positive integers satisfying $s+t \leqslant \kappa-1$. If G is a κ-connected graph, and if the independence number α of G satisfies $\alpha \leqslant \kappa-s-t$, then G is strongly (s, t)-Hamiltonian.*

In fact, it can easily be shown that it suffices to prove the conjecture for $s = 0$, and that a proof similar to that of Theorem 5.1 works if one can first prove the following conjecture:

Conjecture B. *Let $t \leqslant \kappa-1$. If G is a κ-connected graph, and if $\alpha \leqslant \kappa-t$, then every set of t disjoint edges is contained in a circuit of G.*

This second conjecture is a special case of the following conjecture of

Lovász [**45**], on which some progress has been made by Thomassen [**61**] and Woodall [**67**]:

Conjecture C. *If G is a κ-connected graph which is κ+1-edge-connected, then every set of κ vertex-disjoint edges is contained in a circuit of G.*

Sometimes the natural generalizations turn out to be false. For example, one can define a "Hamiltonian-connected digraph" to be a digraph in which there exists a Hamiltonian directed path from any vertex to any other. Ghouila-Houri (1960) has shown that there exist strongly 2-connected digraphs of order p satisfying $\rho(v) \geqslant p+1$ for every vertex v, but with the property that there is no directed path joining some pairs of vertices. However, generalizations with stronger hypotheses have been obtained by Overbeck-Larish [**53**].

We conclude this section by noting that $(p-1)$-edge-Hamiltonian graphs have been characterized. These graphs are called **randomly-Hamiltonian graphs**, and can be described as those graphs in which every path can be completed to a Hamiltonian circuit. Their characterization was carried out by Chartrand and Kronk (1969), and is as follows:

Theorem 9.3. *A graph G is randomly-Hamiltonian if and only if G is either a complete graph, a circuit graph, or a regular complete bipartite graph.* ||

Similar results have been obtained by Thomassen [**59**], who has characterized those graphs in which every path can be extended to a Hamiltonian path, and by Chartrand, Kronk and Lick (1969), who have characterized randomly-Hamiltonian digraphs.

10. Hypohamiltonian Graphs

A graph G is **hypohamiltonian** if G is not Hamiltonian, but the vertex-deleted subgraph $G-v$ is Hamiltonian for every vertex v. For example, the Petersen graph is hypohamiltonian, and is in fact the smallest such graph. The problem of determining those values of p for which a hypohamiltonian graph of order p exists is almost solved, and we summarize the results known in the following theorem (see [**24**], [**58**] and [**60**]):

Theorem 10.1. (*i*) *There exist hypohamiltonian graphs of order p if and only if* $p = 10$ *or* 13, *or* $p \geqslant 15$ (*with the possible exception of* $p = 17$);

(*ii*) *there exist trivalent hypohamiltonian graphs of order p if and only if p is even and* $p = 10$ *or* $p \geqslant 18$ (*with the possible exception of* $p = 24$ *and* $p = 32$);

(*iii*) *there exist planar hypohamiltonian graphs, and hypohamiltonian graphs containing a triangle.* ||

Many constructions for hypohamiltonian graphs have been described, and we now present a construction due to Thomassen [58] which creates new ones from old:

Let G_1 and G_2 be disjoint hypohamiltonian graphs, and suppose that G_1 has a vertex z_1 of valency 3, and that G_2 has a vertex z_2 of valency 3. Moreover, for $i = 1, 2$, let u_i, v_i and w_i be the vertices adjacent to z_i in G_i. We can suppose that G_i does not contain the edges u_iv_i, u_iw_i and v_iw_i, since if G_1 contains the edge u_1v_1 (for example), as $G_1 - u_1$ is Hamiltonian, therefore there exists a Hamiltonian path in $G_1 - u_1 - z_1$ joining v_1 to w_1; by taking this Hamiltonian path, together with the edges w_1z_1, z_1u_1 and u_1v_1, we get a Hamiltonian circuit for G_1, contradicting the fact that G_1 is not Hamiltonian.

Now let G be the graph obtained from G_1 and G_2 by deleting the vertices z_1 and z_2, and identifying the pairs of vertices u_1 and u_2, v_1 and v_2, and w_1 and w_2 (see Fig. 17); we shall call the identified vertices u, v and w, respectively.

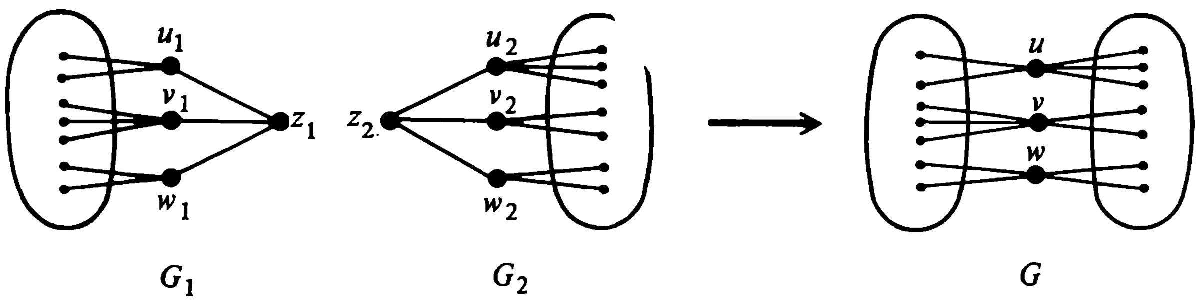


Fig. 17

Theorem 10.2. *The graph G just constructed is hypohamiltonian.*

Proof. We first prove that G is not Hamiltonian. If this is not the case, then G contains a Hamiltonian circuit, and this circuit must pass through u, v and w. The circuit must therefore be the union of a uv-path, a vw-path, and a wu-path, and two of these paths must be contained in $G_1 - z_1$ or $G_2 - z_2$. Without loss of generality, we may assume that the uv-path and the vw-path are contained in $G_1 - z_1$. The union of these two paths is a Hamiltonian path in $G_1 - z_1$ from u_1 to w_1, and by adding z_1 and the edges z_1u_1 and z_1w_1 we obtain a Hamiltonian circuit in G_1, which is a contradiction. So G is not Hamiltonian.

We now prove that if y is any vertex of G, then $G - y$ is Hamiltonian. We may assume without loss of generality that y belongs to $G_1 - z_1$. Since $G_1 - y$ is Hamiltonian, $G_1 - z_1 - y$ has a Hamiltonian path P_1 joining two of the vertices u_1, v_1 and w_1 (u_1 and v_1, say). But $G_2 - w_2$ is also Hamiltonian, and so $G_2 - z_2 - w_2$ has a Hamiltonian path P_2 joining u_2 and v_2. Then $P_1 \cup P_2$ is the required Hamiltonian circuit of $G - y$. This completes the proof. ‖

Now if G_1 contains a vertex y_1 of valency 3 which is not adjacent to z_1, and if G_2 contains a vertex y_2 of valency 3 which is not adjacent to z_2, then G contains two non-adjacent vertices of valency 3. So if there exist two hypohamiltonian graphs (of orders p_1 and p_2, say) with two non-adjacent vertices of valency 3, then there exists a hypohamiltonian graph of order p_1+p_2-5 with two non-adjacent vertices of valency 3. For example, by starting with the Petersen graph, we can construct hypohamiltonian graphs of order $5k$, for every $k \geqslant 2$. Similarly, the existence of hypohamiltonian graphs of orders 10, 13, 16 and 22 with the above property implies the existence of hypohamiltonian graphs of all orders $p \geqslant 13$, except possibly 14, 17 and 19.

A connection between hypohamiltonian graphs and the graphs in the previous section is summarized in the following theorem:

Theorem 10.3. *If there exists a hypohamiltonian graph of order p, then there exists a Hamiltonian-connected graph of order $p+1$ which is not 1-Hamiltonian. Furthermore, if the hypohamiltonian graph contains a vertex of valency 3, then there exists a 1-Hamiltonian graph of order $p-1$ which is not 1-edge-Hamiltonian.*

Proof. Let G be a hypohamiltonian graph, and let G' be the graph obtained from G by adding a new vertex u joined to all the vertices of G. Then G' is Hamiltonian-connected, but not 1-Hamiltonian. To see this, let v and w be vertices of G', and suppose that both of them belong to G (the case in which one of them is equal to u is similar); then in $G-v$ there is a Hamiltonian path from w to w' (say), which together with the edges $w'u$ and uv gives a Hamiltonian path from w to v.

Now let z be a vertex of valency 3, and let $\bar{u}$, $\bar{v}$ and $\bar{w}$ be the vertices adjacent to z. Let G' be the graph obtained from G by deleting the vertex z and adding the edges $\bar{u}\bar{v}$, $\bar{u}\bar{w}$ and $\bar{v}\bar{w}$ (which are not in G, as we proved just before Theorem 10.2). Now G' is 1-Hamiltonian, since if y is a vertex of G', then $G-y$ has a Hamiltonian circuit containing two of the three edges zu, zv and zw, and so $G'-y$ is Hamiltonian. However, G' is not 1-edge-Hamiltonian, since there is no Hamiltonian path from u to v. This completes the proof. ||

One can similarly define a graph G to be **hypotraceable** if G is not traceable, but the vertex-deleted subgraph $G-v$ is traceable for every vertex v. The best results in this direction have been obtained by Thomassen **[58]**, **[60]**, who has proved the existence of hypotraceable graphs of order p, for $p = 34$, $p = 37$ and $p \geqslant 39$. However, several problems remain open; for example, Grünbaum has asked whether there exists a non-Hamiltonian graph G with the property that every vertex-deleted subgraph $G-v$ is non-Hamiltonian, but every "two-vertex-deleted subgraph" $G-v-w$ is Hamiltonian.

Finally, we remark that Fouquet and Jolivet [30] have proved that if $p \geqslant 6$, then there exists a hypohamiltonian digraph (defined in the obvious way) of order p. However, if we impose the extra restriction that pairs of opposite arcs are not allowed, then the set of numbers p for which such a digraph exists is unknown, although Thomassen has obtained partial results.

11. Some Miscellaneous Results

There are a great number of articles in which Hamiltonian paths and circuits play an important role, but whose results do not fit easily into any of the preceding sections. In this section we shall mention some of these results.

Hamiltonian Decompositions

We shall say that a graph G can be decomposed into Hamiltonian circuits if the edge-set of G can be partitioned into Hamiltonian circuits; similarly, we say that G can be decomposed into Hamiltonian paths if the edge-set of G can be partitioned into Hamiltonian paths (see [8] for a survey on this topic). The best-known result of this type is the following:

Theorem 11.1. *(i) if p is even, K_p can be decomposed into $\frac{1}{2}p$ Hamiltonian paths;*

(ii) if p is odd, K_p can be decomposed into $\frac{1}{2}(p-1)$ Hamiltonian circuits.

Proof. (*i*) Let the vertices of K_p be $v_1, v_2, \ldots, v_p$. Then the $\frac{1}{2}p$ Hamiltonian paths required are

$$v_{1+i}, v_{p+i}, v_{2+i}, v_{p+i-1}, \ldots, v_{j+i+1}, v_{p+i-j}, \ldots, v_{\frac{1}{2}p+i}, v_{\frac{1}{2}p+i+1},$$

for $i = 1, 2, \ldots, \frac{1}{2}p$, where the subscripts are all taken modulo p.

(*ii*) The Hamiltonian circuits required are obtained from the paths of part (*i*) by joining an extra vertex to each end of these paths. ‖

One can also use a similar method to decompose K_p into $\frac{1}{2}(p-2)$ Hamiltonian circuits and a 1-factor, if p is even.

The corresponding results for digraphs are more difficult to prove, and have been obtained only recently by Tillson [63]:

Theorem 11.2. *If $p \neq 4$ or 6, then the complete symmetric digraph of order p can be decomposed into $p-1$ Hamiltonian directed circuits.* ‖

However, many similar problems in this area are unsolved, including the following conjectures of Jackson [38], and Kelly:

Conjecture A. *If G is a regular ρ-valent graph of order p, where* $\rho \geqslant \frac{1}{2}(p-1)$, *then G has* $[\frac{1}{2}p]$ *edge-disjoint Hamiltonian circuits.*

Conjecture B. *Every regular tournament can be decomposed into Hamiltonian directed circuits.*

Enumeration Problems

How many Hamiltonian circuits does a given graph possess? This question and related problems seem to be very difficult, and results are few and far between. For example, the number $h(n)$ of Hamiltonian circuits in the n-cube is not known, except for the values $h(2) = 1$, $h(3) = 6$, $h(4) = 1344$, and $h(5) = 906\ 545\ 760$ (probably); for references and bounds on $h(n)$, see **[25]**.

Other results are concerned with the parity of the number of Hamiltonian circuits in a graph. Results of this kind include the following, due respectively to Smith (1946), and Kotzig (1966):

Theorem 11.3. *Let G be a trivalent graph. Then*

(*i*) *the number of Hamiltonian circuits of G containing a given edge is even*;

(*ii*) *if G is bipartite, then G contains an even number of Hamiltonian circuits.* ||

For a proof of these results, see (for example) Thomason **[57]** (not to be confused with Thomassen!) who has also proved the following result (see Section 8 of Chapter 5):

Theorem 11.4. *If G can be expressed as the union of two edge-disjoint Hamiltonian circuits, then G can also be expressed as the union of two other edge-disjoint Hamiltonian circuits.* ||

Results on the number of Hamiltonian circuits in a planar graph have also been obtained by various authors.

Vertex-transitive Graphs

A graph is **vertex-transitive** if, given any two vertices v and w of G there is an automorphism of G which maps v to w. Lovász has formulated the following conjecture:

Conjecture. *Every connected vertex-transitive graph is traceable.*

A related conjecture was made by Thomassen. While studying connected vertex-transitive graphs, he found only four which fail to be Hamiltonian—

namely, the Petersen graph, the Coxeter graph (see, for example, [**17**, p. 241]), and the graphs obtained by replacing each vertex of these graphs by a triangle, and on the strength of this, he formulated the following conjecture:

Conjecture. *There are only a finite number of connected vertex-transitive graphs which are non-Hamiltonian.*

Comparability Graphs

If D is any digraph, then the comparability graph of D is the (undirected) graph G whose vertex-set is the same as that of D, and whose edges join those pairs of vertices which are connected by a directed path in D. If D is a tree in which every arc is directed towards some root-vertex, then it can be shown that a graph G is the comparability graph of D only if for every path u, v, w, x, y in G, either uw or vx is an edge. Using the concepts of a "good valuation" and a "pseudo-valuation", Arditti and Cori have completely characterized those trees whose comparability graph is Hamiltonian; details of their results may be found in [**2**].

12. Generalizations

When a graph is not Hamiltonian, it is natural to ask "how far" it is from being Hamiltonian. This has given rise to many possible generalizations, several of which are extensions of the results of Section 4. We shall mention some of these generalizations here.

The Circumference of a Graph

Here one asks for the length of the longest circuit in the graph. When this circuit includes every vertex of the graph, we have a Hamiltonian circuit. For results on the circumference of a graph, see [7], [**12**] and [**66**].

Pseudo-Hamiltonian Circuits

If G is a graph of order p, a "pseudo-Hamiltonian circuit", or "Hamiltonian walk", is a closed walk (not necessarily a circuit) of minimum length which includes every vertex of G. Clearly, the length of such a walk is at least p, and is equal to p if and only if G is Hamiltonian. It is easy to see that if G contains a circuit of length c, then there is a pseudo-Hamiltonian circuit whose length is at most $2p-c$; for further details, see [7] and [**39**].

Vertex-partitions

Here the problem is to determine the minimum number of vertex-disjoint circuits (or paths) which cover all of the vertices of the graph G, and therefore partition its vertex-set. When this minimum number is 1, the graph is Hamiltonian. If G is not Hamiltonian, then the minimum number of paths which partition the vertex-set is equal to the minimum number of edges that must be added to G to make it Hamiltonian; for further details, see **[39]** and **[56]**.

Spanning Trees

We can also investigate spanning trees of a graph, and ask for the spanning trees with the minimum number of vertices of valency 1. If this minimum number is 2, the graph is traceable; for further details, see **[42]**.

Several theorems on Hamiltonian graphs have been extended to these concepts, although many problems remain open. We conclude with a theorem generalizing Ore's theorem (Corollary 4.2) which includes all of the above generalizations, and some related conjectures:

Theorem 12.1. *Let G be a connected graph of order p, and let c be a number satisfying $c \leqslant p$. If $\rho(v)+\rho(w) \geqslant c$ for each pair of non-adjacent vertices v and w in G, then*

(*i*) *the circumference of G is at least $\frac{1}{2}c$;*

(*ii*) *if G is 2-connected, the circumference of G is at least c;*

(*iii*) *G contains a pseudo-Hamiltonian circuit whose length is at most $2p-c$;*

(*iv*) *the vertices of G can be partitioned into at most $p-c+1$ circuits, or (if $c < p$) at most $p-c$ paths;*

(*v*) *if $c < p$, there exists a spanning tree with at most $p-c+1$ vertices of valency 1;*

(*vi*) *if G has connectivity κ and independence number α, where $\kappa \geqslant \alpha - c$, then G has a spanning tree with at most $c+1$ vertices of valency 1.* ‖

The following conjectures are due, respectively, to Woodall and Jolivet:

Conjecture A. *If G is a 2-connected graph of order p, and if G has at least $k+\frac{1}{2}p$ vertices whose valency is at least k, then the circumference of G is at least $2k$.*

Conjecture B. *If G has connectivity* κ *and independence number* α, *then the circumference of G is at least* $\{\kappa(\alpha+p-\kappa)/\alpha\}$,

Conjecture C. *If* $\kappa \geqslant \alpha - c$, *then G has a spanning tree with at most* $c+1$ *vertices of valency* 1.

Recently S. Win announced a proof of Conjecture C (*Graph Theory Newsletter* 7 (1978)).

References

For general surveys of the topics covered in this chapter, the reader is referred in particular to **[6**, Chapter 10], **[17**, Chapters 4 and 9], **[19]**, **[43]**, **[49]**, **[50]** and **[51]**. Necessary conditions for Hamiltonian graphs are discussed in **[21]**, **[22]** and **[23]**, sufficient conditions on valencies in **[16]** and **[43]**, planar graphs in **[34]**, pancyclic graphs in **[13]** and **[14]**, and strongly Hamiltonian graphs in **[65]**. Recently A. Bondy has written an interesting survey: "Hamilton cycles in graphs and digraphs" presented at the Ninth Southeastern Conference on Combinatorics, 1978.

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