SOME HAMILTONIAN COUNTEREXAMPLES

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In this article we survey some interesting examples and counterexamples concerning hamiltonian graphs and related concepts. The discussion is divided into three areas: necessary or sufficient conditions, planar graphs, and hamiltonianlike conditions. Throughout we shall use the terminology of Harary [16]. In particular, a graph is hamiltonian if it possesses a cycle which passes through each of its points exactly once. The number of points and lines of a graph are denoted by p and q, respectively. We assume throughout that $p \geq 3$ and that all graphs are connected. The degree of a point v is denoted by $d(v)$ and the minimum degree of the points of G by $\delta(G)$. The degree sequence of a graph is written in nondecreasing order, $d_1 \leq d_2 \leq \cdots \leq d_p$. Let H be a graph of order p which contains the graph G such that $d_H(u) + d_H(v) < p$, for all $uv \notin E(H)$. If H_1 and H_2 are such graphs, then so is $H_1 \cap H_2$. Thus, there exists a unique smallest graph H with the above property. We call this graph the closure of G and denote it by $\mathrm{cl}(G)$. One can thus obtain $\mathrm{cl}(G)$ from G by joining pairs of nonadjacent points of G with degree sum at least p until no such pair remains.

SECTION 1

Most of the sufficient conditions for a graph to be hamiltonian force the graph to have many lines relative to the number of its points. A sequence of successively stronger theorems giving sufficient conditions for a graph to be hamiltonian were developed beginning in 1952. Each of the following is sufficient for a graph to be hamiltonian:

- S1. $\delta(G) \geq p/2$ [9].
- S2. u nonadjacent to v implies $d(u) + d(v) \geq p$ [27].
- S3. For every n, $1 \leq n \leq (p-1)/2$, the number of points of degree not exceeding n is less than n and, for odd p, the number of points of degree at most $(p-1)/2$ does not exceed $(p-1)/2$ [28].
- S4. $d_i \leq i$ and $d_j \leq j$ imply $d_i + d_j \geq p$ [1].
- S5. $d_i \leq i < p/2$ implies $d_{p-i} \geq p - i$ [6].
- S6. If the points of G are $v_1, \ldots, v_p$, let there be no i, j with $i < j$, $v_iv_j \notin E(G)$, $d(v_i) \leq i$, $d(v_j) \leq j - 1$, and $d(v_i) + d(v_j) \leq p - 1$, with $i + j \geq p$ [20].
- S7. $\mathrm{cl}(G) = K_p$ [3].

It is interesting to note that none of these conditions is strong enough to prove that the simple 5-cycle C_5 is hamiltonian.

One of the few useful necessary conditions for hamiltonicity may be phrased in terms of the toughness of a graph. If $k(G)$ denotes the number of connected components of the graph G, we define G to be t-tough if $k(G - S) > 1$ implies that $|S| \geq t \cdot k(G - S)$ for all sets S of points of G. Chvátal [7] has proved that every hamiltonian graph is 1-tough. That 1-toughness is not sufficient for hamiltonicity is shown by the graph of FIGURE 1.

If G is not complete, the largest t for which G is t-tough is called the toughness of G and is denoted $t(G)$. Chvátal conjectured that every graph G with $t(G) > 3/2$

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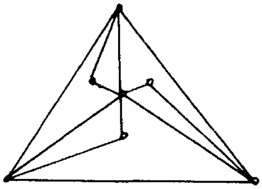

FIGURE 1. A 1-tough nonhamiltonian graph.

is hamiltonian. That this conjecture could not be improved to include $t = 3/2$ is shown by the nonhamiltonian graph of FIGURE 2b. This graph is the inflation of the Petersen graph which is depicted in FIGURE 2a. The inflation G^* of a graph G is the graph whose points are all ordered pairs (v, x), where x is a line of G and v is an endpoint of x, with two points of G^* adjacent if and only if they differ in exactly one coordinate. Although true for planar graphs, since a graph with $t > 3/2$ is 4-connected (see Section 2), Chvátal's conjecture has recently been disproved by C. Thomassen [34].

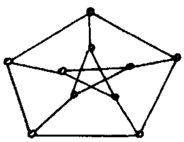

FIGURE 2(a). The Petersen graph.

A graph which is obtained in a manner similar to that of the inflation of the Petersen graph is due to Meredith [25] and appears in FIGURE 3. This graph was obtained as a counterexample to the following conjecture of Nash-Williams [26]: If G is 4-connected and 4-regular, then G is hamiltonian. If one shrinks each of the $K_{3,4}$'s to a point, it is easy to show that the resulting multigraph is nonhamiltonian. This in turn implies that the original graph is nonhamiltonian.

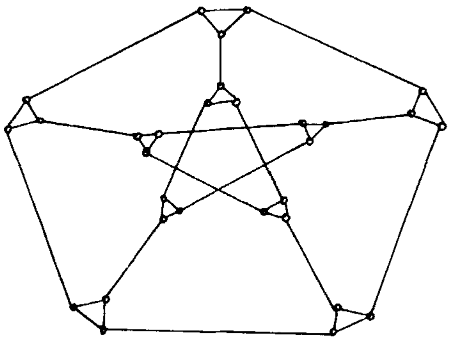

FIGURE 2(b). A nonhamiltonian graph with $t = 3/2$.

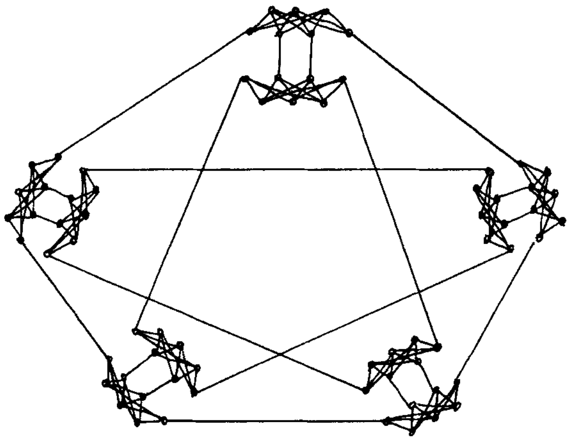

FIGURE 3. A 4-connected, 4-regular, nonhamiltonian graph.

SECTION 2

Planar hamiltonian graphs historically have been the object of intensive investigation because of their relation to the Four Color Problem. More recently they have found application to the structure of organic compounds and to linear programming algorithms.

A cyclic cutset L of a 3-connected graph G is a set of lines of G such that $G - L$ has two components each of which contains a cycle. The cyclic connectivity of G, $c\lambda(G)$, is the minimum cardinality of all the cyclic cutsets of G. The graph G is cyclically n-connected if $c\lambda(G) \geq n$. We define a graph G to be polyhedral if it is cubic, planar, and 3-connected.

A proof of the Four Color Theorem could be based on any one of the following statements, if any were true:

P1. Every polyhedral graph is hamiltonian.

P2. Every polyhedral cyclically 4-connected graph is hamiltonian.

P3. Every polyhedral cyclically 5-connected graph is hamiltonian.

Unfortunately, for the Four Color Problem, they are all false. The counterexamples that have been constructed to prove them false, however, are very interesting graphs.

The first counterexample to P1 was the forty-six-point graph of FIGURE 4 constructed by Tutte [35] and is now known as the Tutte graph. For later reference we

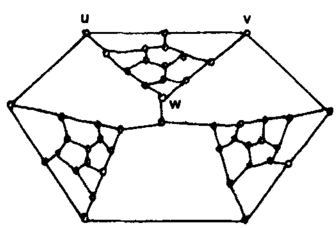

FIGURE 4. The Tutte graph: a nonhamiltonian polyhedral graph.

shall refer to the fifteen-point triangular configuration with vertices u, v, and w as the Tutte configuration.

Since 1946 a great deal of effort has gone into finding the smallest nonhamiltonian polyhedral graph. It has been shown [14] that all polyhedral graphs with up to twenty points are hamiltonian. The smallest known nonhamiltonian polyhedral graph, discovered independently by Lederberg [21], Barnette, and Bosak [5], has thirty-eight points and is depicted in FIGURE 5. Note the use of the Tutte configuration.

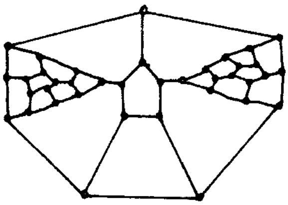

FIGURE 5. The smallest-known nonhamiltonian poly hedral graph.

A variation of statement P1 has given rise to a very interesting graph due to J. D. Horton [4]. Tutte conjectured that every bipartite cubic 3-connected graph is hamiltonian. Horton's graph (FIGURE 6) shows this to be false.

Still other variations on statement P1 may be obtained as follows. Tutte [36] proved that every planar 4-connected graph is hamiltonian. If we now consider 3-connected planar graphs, we can ask what additional condition on these graphs will force them to be hamiltonian? The hypothesis of statement P1 adds the condition that G be regular of degree 3, which is not enough to force G to be hamiltonian. Suppose we require that G be regular of degree 4, or regular of degree 5 (if G were regular of degree 6 or greater, then G would not be planar). Sachs [29] showed that neither condition forces G to be hamiltonian. Recently Zaks [39] simplified Sachs' constructions by finding a planar 3-connected, 4-regular, nonhamiltonian graph with 114 points and a planar 3-connected, 5-regular, nonhamiltonian graph with 228 points. We shall omit these graphs but mention that they are derived from the graph G of FIGURE 5 by replacing the edges of a 1-factor of G by certain configurations.

The first counterexample to statement P2 was constructed by Tutte [37] and has sixty points. The smallest known counterexample to P2, which has forty-two

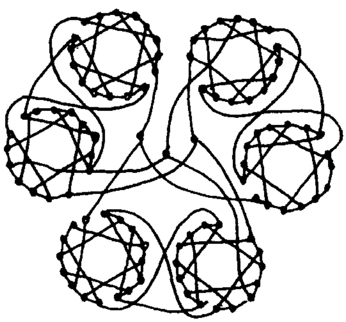

FIGURE 6. A bipartite 3-connected nonhamiltonian graph.

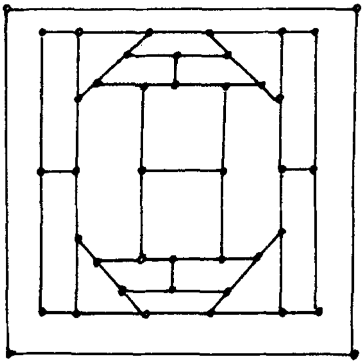

FIGURE 7. The smallest-known polyhedral cyclically 4-connected nonhamiltonian graph.

points, is shown in FIGURE 7 [15]. The first counterexample to statement P3 was constructed by Walther [38] and has 162 points. The graph of FIGURE 8, which appears in [15], is due independently to Tutte and Grinberg, has forty-four points, and is the smallest known counterexample to P3.

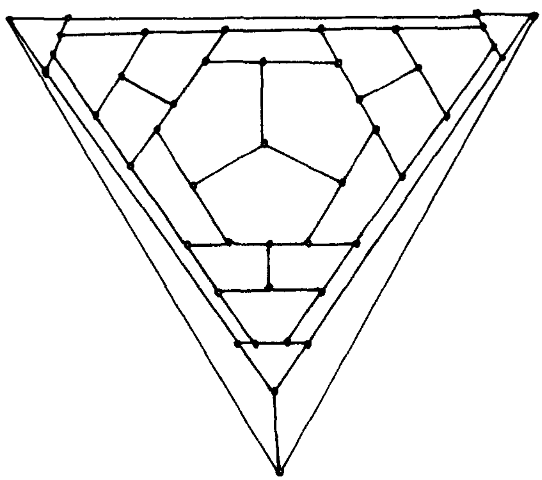

FIGURE 8. The smallest-known polyhedral cyclically 5-connected nonhamiltonian graph.

SECTION 3

We shall first consider some conditions weaker than that of a graph being hamiltonian. The nth power G^n of G has the same set of points as G with two points adjacent if and only if the distance between them in G is n or less. Sekanina [31] and Karaganis [19] have proved that every connected graph G has G^3 hamiltonian. Fleischner [12] then showed that if G is 2-connected, then G^2 is hamiltonian.

(In fact, in this case much more is true of G^2. Faudree and Schelp [11] have shown that G^2 is strongly path-connected, i.e., between each pair of distinct points u and v of G^2 there exists a path in G^2 of every possible length greater than or equal to the distance between u and v in G^2.) Fleischner's theorem cannot be improved. FIGURE 9a shows a tree whose square is nonhamiltonian and FIGURE 9b shows a graph, due to Fleischner and Kronk [13], which is bridgeless and has a nonhamiltonian square.

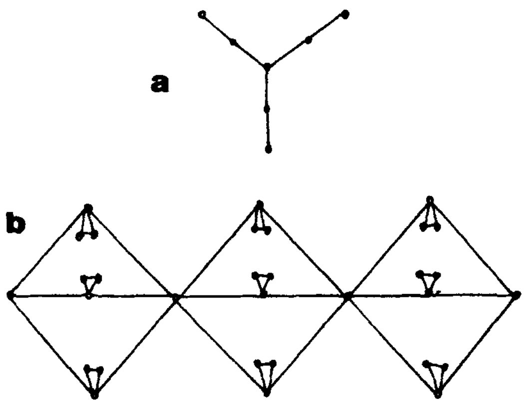

FIGURE 9. (a) A tree whose square is nonhamiltonian. (b) A bridgeless graph whose square is nonhamiltonian.

A graph is traceable if it has a hamiltonian path. Balinski conjectured that every polyhedral graph is traceable. The graph in FIGURE 10, constructed by T. A. Brown [14], shows this to be false. In each triangle T of FIGURE 10 is a copy of the Tutte configuration placed so that the points u, v, and w of FIGURE 4 get mapped to the correspondingly labelled points.

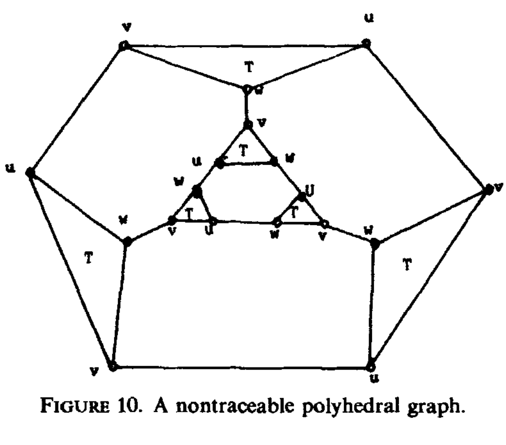

FIGURE 10. A nontraceable polyhedral graph.

A nonhamiltonian (nontraceable) graph G is hypohamiltonian (hypotraceable) if $G - v$ is hamiltonian (traceable) for every point v of G. The following theorem [17, 23, 32, 10, 8] establishes the existence of hypohamiltonian graphs of almost all orders p.

THEOREM. There exist no hypohamiltonian graphs on points $p < 10$, $p = 11$, 12, or 14. For $p = 10$, 13, and $p \geq 15$, except possibly $p = 17$, there exists a hypohamiltonian graph of order p.

The smallest hypohamiltonian graph is the Petersen graph (FIGURE 2a).

Less is known about hypotraceable graphs. We have the following theorem of Thomassen [32, 34].

THEOREM. There exist hypotraceable graphs of order p for $p = 34$, 37 and all $p \geq 39$.

It is not known if hypotraceable graphs of orders 35, 36, or 38 exist, nor is the order of the smallest hypotraceable graph known. The smallest hypotraceable graph constructed to date is depicted in FIGURE 11.

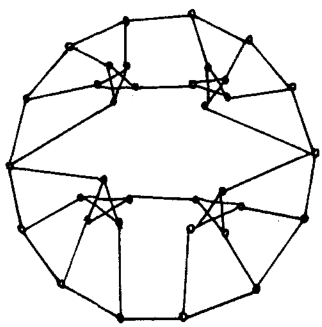

FIGURE 11. The smallest-known hypotraceable graph.

The only condition stronger than hamiltonicity that will be mentioned is that of a graph being pancyclic. The graph G is pancyclic if it contains cycles of all lengths n, $3 \leq n \leq p$. Many of the conditions which are sufficient for a graph to be hamiltonian are also sufficient for a graph to be pancyclic or bipartite. For example, Ore's condition S2 implies that G is pancyclic or bipartite [2]; Chvátal's condition S5 implies that G is pancyclic or bipartite [30]; G 2-connected implies that G^2 is pancyclic [18]. Such results led Bondy to make the metaconjecture that almost all conditions that imply that G is hamiltonian also imply that G is pancyclic or bipartite.

The last examples we present are cases where the metaconjecture is false. As mentioned previously, every planar 4-connected graph is hamiltonian. Malkevitch [24] has constructed the graph of FIGURE 12 which is planar and 4-connected but which is not pancyclic since it does not contain a 4-cycle.

The graph of FIGURE 13, due to C. Thomassen, was communicated to the author by J. A. Bondy. It is an example of a graph with a complete closure but which is not pancyclic since it contains no triangles. Hence, condition S7 does not imply that G is pancyclic.

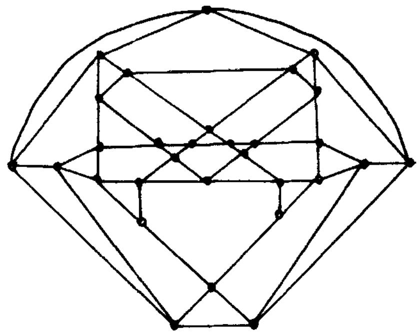

FIGURE 12. A planar 4-connected nonpancyclic graph.

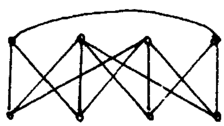

FIGURE 13. A nonpancyclic graph with a complete closure.

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