NORTH-HOLLAND

Hamilton Cycles and Eigenvalues of Graphs

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Dedicated to J. J. Seidel

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ABSTRACT

We prove some results concerning necessary conditions for a graph to be Hamiltonian in terms of eigenvalues of certain matrices associated with the graph. As an example, we show how the results give an easy algebraic proof of the nonexistence of a Hamilton cycle in two graphs, one of them being the Petersen graph.

1. INTRODUCTION

There exist many results that show a relationship between eigenvalues of a graph and structural properties of the graph. By eigenvalues of a graph, we mean the eigenvalues of a certain matrix derived from the graph, where we must specify how the matrix is derived from the graph in order for this information to make sense. In this note, we prove some results that connect the existence of a Hamilton cycle in the graph and bounds on the eigenvalues of the graph. A first theorem in this direction was given in Mohar [9], but the condition in [9] only holds for regular graphs and also involves some rather complicated considerations. In contrast, our results hold for general, nonregular graphs. Also, the proofs are very easy and almost immediately follow from well-known properties of the spectra of graphs, but the conditions are

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powerful enough to imply, using eigenvalues only, that the Petersen graph is non-Hamiltonian. The theorems with proofs and some corollaries are given in the next section. At the end of the paper we show that the result in [9] is a consequence of our theorems.

In the remainder of this Introduction we give some definitions and notation. We use Bondy and Murty [1] as our main source and always assume that a graph is simple and finite.

For a graph G we use A_G to denote the *adjacency matrix of* G and D_G to denote the diagonal matrix with the degrees of the vertices of G on the diagonal. In these definitions and in the ones in the remainder of the paper, we will assume that the rows and columns of the matrices are labeled with the vertices of the graph in a certain fixed order. Set

$$L_G = D_G - A_G \quad \text{and} \quad Q_G = D_G + A_G.$$

The matrix L_G is well studied (see, e.g., Cvetković, Doob, and Sachs [2], Merris [7], or Mohar [8]) and appears under several names. We will follow what appears to be the majority and call L_G the *Laplacian of* G, a name due to its connection with the Laplacian operator in the theory of Riemannian manifolds.

The matrix Q_G seems to be less well known. The monograph [2] gives only two references, both written in Cyrillic characters. Very recently the matrix also appeared in Desai and Rao [3]. It appears that there is a strong connection between the eigenvalues of Q_G and the eigenvalues of the adjacency matrix of the line graph of G. [The line graph of a graph G, sometimes called edge graph or derived graph, is the graph with vertex set $E(G)$ in which two vertices are joined if and only if they have a common end vertex in G.]

If M is an $n \times n$ symmetric matrix, then we denote the eigenvalues of M by $\lambda_i(M)$, $i = 1, \dots, n$, with the order $\lambda_1(M) \leqslant \lambda_2(M) \leqslant \cdots \leqslant \lambda_n(M)$.

We use C_n to denote the cycle on n vertices.

2. RESULTS

Our main result is the following theorem.

THEOREM 1. *Let G be a graph on n vertices. If G contains a Hamilton cycle, then for $i = 1, \dots, n$,*

$$\lambda_i(L_{C_n}) \leqslant \lambda_i(L_G) \quad \textit{and} \quad \lambda_i(Q_{C_n}) \leqslant \lambda_i(Q_G).$$

In fact, we will prove the following theorem, which is slightly stronger for graphs with only a small number of edges.

THEOREM 1′. *Let G be a graph on n vertices and m edges. Suppose G contains a Hamilton cycle. Then for $i = 1, \dots, n$,*

$$\lambda_i(L_{C_n}) \leqslant \lambda_i(L_G) \quad \text{and} \quad \lambda_i(Q_{C_n}) \leqslant \lambda_i(Q_G).$$

In addition, if $m < 2n$, then for $i = m - n + 1, \dots, n$, we have

$$\lambda_{i-m+n}(L_G) \leqslant \lambda_i(L_{C_n}) \leqslant \lambda_i(Q_G)$$

and

$$\lambda_{i-m+n}(Q_G) > \lambda_i(Q_{C_n}) \leqslant \lambda_i(L_G).$$

Theorems 1 and 1′ follow immediately from the following lemma, using the obvious fact that a Hamilton cycle in a graph G can be considered as an edge-deleted subgraph of G isomorphic to C_n.

LEMMA 2. *Let G be a graph on n vertices and let H be a subgraph of G obtained by deleting an edge in G. Then*

$$0 \leqslant \lambda_1(L_H) \leqslant \lambda_1(L_G) \leqslant \lambda_2(L_H) \leqslant \lambda_2(L_G) \leqslant \cdots$$
$$\leqslant \lambda_{n-1}(L_G) \leqslant \lambda_n(L_H) \leqslant \lambda_n(L_G)$$

and

$$0 \leqslant \lambda_1(Q_H) \leqslant \lambda_1(Q_G) \leqslant \lambda_2(Q_H) \leqslant \lambda_2(Q_G) \leqslant \cdots$$
$$\leqslant \lambda_{n-1}(Q_G) \leqslant \lambda_n(Q_H) \leqslant \lambda_n(Q_G).$$

Proof. The properties in the lemma are well known to hold for the Laplacian of a graph (see, e.g., [8]). For completeness, we give the proof for both L_G and Q_G.

The *incidence matrix of* G, denoted P_G, is the $|V(G)| \times |E(G)|$ matrix with entries

$$(P_G)_{ue} = \begin{cases} 1, & \text{if } u \text{ and } e \text{ are incident,} \\ 0, & \text{if } u \text{ and } e \text{ are not incident,} \end{cases} \qquad u \in V(G), e \in E(G).$$

Next we choose an orientation on G, i.e., for each $e \in E(G)$ we choose one of its end vertices as the initial vertex and the other end vertex as the terminal vertex. The *oriented incidence matrix* with respect to the chosen orientation is the $|V(G)| \times |E(G)|$ matrix K_G with entries

$$(K_G)_{ue} = \begin{cases} 1, & \text{if } u \text{ is the terminal vertex of } e, \\ -1, & \text{if } u \text{ is the initial vertex of } e, \\ 0, & \text{if } u \text{ and } e \text{ are not incident,} \end{cases} \qquad u \in V(G), e \in E(G).$$

Now it is easy to check that

$$L_G = K_G K_G^{\mathsf{T}} \quad \text{and} \quad Q_G = P_G P_G^{\mathsf{T}},$$

where N^{T} denotes the transpose of the matrix N. Note that K_G depends on the chosen orientation on G, whereas L_G does not.

It is well known that if N is a matrix, then both NN^{T} and $N^{\mathsf{T}}N$ are symmetric matrices with only nonnegative eigenvalues. Moreover, apart from the multiplicity of the eigenvalue 0, the spectra of NN^{T} and $N^{\mathsf{T}}N$ coincide (see, e.g., Godsil [4, p. 186]). In particular, the positive eigenvalues of L_G are the same as the positive eigenvalues of $K_G^{\mathsf{T}} K_G$ and a similar relation holds for Q_G and $P_G^{\mathsf{T}} P_G$.

Let H be a subgraph of G obtained by deleting an edge from G. Then the matrix $K_H^{\mathsf{T}} K_H$ is obtained from $K_G^{\mathsf{T}} K_G$ be deleting the row and column corresponding to the deleted edge, and $P_H^{\mathsf{T}} P_H$ is obtained from $P_G^{\mathsf{T}} P_G$ in the similar way. Using the interlacing properties of principal submatrices of symmetric matrices (see, e.g., Van Lint and Wilson [5, p. 396]) and the observations above, this proves the lemma. ∎

Note that if G is a graph with n vertices and m edges, then the matrix $P_G^{\mathsf{T}} P_G$ defined in the proof above is equal to the matrix $2I_m + A_{L(G)}$, where I_m is the $m \times m$ identity matrix and $L(G)$ denotes the line graph of G. This means that a real number λ is a nonnegative eigenvalue of Q_G if and only if $-2 + \lambda$ is an eigenvalue of $A_{L(G)}$ greater than -2.

For regular graphs we can give the following form of Theorem 1 in terms of the eigenvalues of the adjacency matrix.

COROLLARY 3. *Let G be a k-regular graph on n vertices. If G contains a Hamilton cycle, then for $i = 1, \ldots, n$,*

$$\lambda_i(A_G) - (k-2) \leqslant \lambda_i(A_{C_n}) \leqslant \lambda_i(A_G) + (k-2).$$

Proof. If G is a k-regular graph, then we have $L_G = kI_n - A_G$ and $Q_G = kI_n + A_G$; hence, $\lambda_i(L_G) = k - \lambda_i(A_G)$ and $\lambda_i(Q_G) = k + \lambda_{n-i+1}(A_G)$, for $i = 1, \ldots, n$. Of course, this also means $\lambda_i(L_{C_n}) = 2 - \lambda_i(A_{C_n})$ and $\lambda_i(Q_{C_n}) = 2 + \lambda_{n-i+1}(A_{C_n})$. Substitution in Theorem 1 gives the result. ∎

The eigenvalues of A_{C_n} are

$$\left\{-2_{[1]}, 2\cos\left(\frac{(n-2)\pi}{n}\right)_{[2]}, 2\cos\left(\frac{(n-4)\pi}{n}\right)_{[2]}, \ldots, 2\cos\left(\frac{2\pi}{n}\right)_{[2]}, 2_{[1]}\right\},$$

if n is even, and

$$\left\{2\cos\left(\frac{(n-1)\pi}{n}\right)_{[2]}, 2\cos\left(\frac{(n-3)\pi}{n}\right)_{[2]}, \ldots, 2\cos\left(\frac{2\pi}{n}\right)_{[2]}, 2_{[1]}\right\},$$

if n is odd. Here $\lambda_{[l]}$ denotes that the eigenvalue λ has multiplicity l. The eigenvalues of the adjacency matrix of the Petersen graph P are $\{-2_{[4]}, 1_{[5]}, 3_{[1]}\}$. This means $\lambda_5(A_P) - 1 = 0 > 2\cos(3\pi/5) = \lambda_5(A_{C_{10}})$ and $\lambda_4(A_{C_{10}}) = 2\cos(3\pi/5) = \frac{1}{2} - \frac{1}{2}\sqrt{5} > -1 = \lambda_4(A_P) + 1$, so by Corollary 3 and the fact that P is 3-regular, the Petersen graph is non-Hamiltonian.

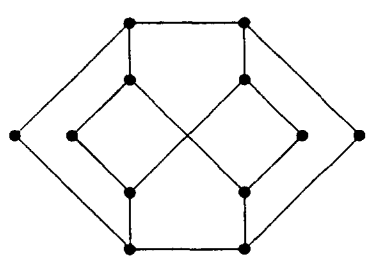

FIG. 1.

An example of a nonregular graph that can be shown to be non-Hamiltonian by Theorem 1 is the graph G depicted in Figure 1. This graph appears in Kratsch, Lehel, and Müller [6] as an example of a bipartite graph that satisfies most known necessary conditions for the existence of a Hamilton cycle (e.g., it is 1-tough and contains a 2-factor), but is non-Hamiltonian. Using a symbolic manipulation package like Maple, we find that the eigenvalues of L_G are $\{0_{[1]}, 1_{[2]}, (\frac{7}{2} - \frac{1}{2}\sqrt{17})_{[1]}, (3 - \sqrt{2})_{[2]}, 3_{[1]}, 4_{[2]}, (3 + \sqrt{2})_{[2]}, (\frac{7}{2} + \frac{1}{2}\sqrt{17})_{[1]}\}$. This means $\lambda_6(L_{C_{12}}) = 2 - 2\cos(\frac{1}{2}\pi) = 2 > 3 - \sqrt{2} = \lambda_6(L_G)$. So by Theorem 1 the graph is non-Hamiltonian.

Finally, we show that the main result in [9] follows from Theorem 1′. In order to do this we need some extra terminology.

We extend our concept of a graph by allowing *free edges*, which are edges with only one end vertex. In this case, we define the degree of a vertex as the number of (ordinary or free) edges incident with the vertex. Also, free edges do not appear in the adjacency matrix of a graph. With these conventions we can define the Laplacian of a graph with free edges as before. By $C_{2n,l}$ we denote the graph obtained from the cycle C_{2n} by adding l free edges to every second vertex on the cycle. If G is a graph, then the *subdivision graph* $S(G)$ is obtained from G by subdividing each edge in G.

COROLLARY 4 (Mohar [9]). *Let G be a k-regular graph of order n. Let $S(G)$ be the subdivision graph of G. If G has a Hamilton cycle, then for $i = 1, \ldots, 2n$,*

$$\lambda_i(L_{S(G)}) \leqslant \lambda_i(L_{C_{2n,k-2}}).$$

Proof. Let $m = \frac{1}{2}nk$ be the number of edges of G. From [9, Lemma 3.1] it follows that if the Laplacian of G has eigenvalues $\lambda_1(L_G) \leqslant \lambda_2(L_G) \leqslant \cdots \leqslant \lambda_n(L_G)$, then the eigenvalues of the Laplacian of $S(G)$ are

$$\lambda_i(L_{S(G)}) = \begin{cases} \frac{1}{2}(k+2) - \frac{1}{2}\sqrt{(k+2)^2 - 4\lambda_i(L_G)}, & i = 1, \ldots, n, \\ 2, & i = n+1, \ldots, m, \\ \frac{1}{2}(k+2) + \frac{1}{2}\sqrt{(k+2)^2 - 4\lambda_{m+n-i+1}(L_G)}, & \\ \qquad i = m+1, \ldots, m+n. & \end{cases}$$

Furthermore, it is proved in [9, Lemma 3.2] that the eigenvalues of the Laplacian of $C_{2n,k-2}$ are $\frac{1}{2}(k+2) + \frac{1}{2}\sqrt{(k-2)^2 + 8 + 8\cos(2\pi j/n)}$ and $\frac{1}{2}(k+2) - \frac{1}{2}\sqrt{(k-2)^2 + 8 + 8\cos(2\pi j/n)}$, for $j = 1, \ldots, n$. Ordering these eigenvalues, we obtain

$$\lambda_i(L_{C_{2n,k-2}})$$

$$= \begin{cases} \frac{1}{2}(k+2) - \frac{1}{2}\sqrt{(k-2)^2 + 8 + 8\cos((2\pi\lfloor i/2 \rfloor)/n)}, \\ \qquad i = 1, \ldots, n, \\ \frac{1}{2}(k+2) + \frac{1}{2}\sqrt{(k-2)^2 + 8 + 8\cos((2\pi\lfloor (2n-i+1)/2 \rfloor)/n)} \\ \qquad i = n+1, \ldots, 2n. \end{cases}$$

Substituting these expressions, it follows that the condition in the corollary is equivalent to

$$2 + 2\cos\left(\frac{2\pi\lfloor i/2 \rfloor}{n}\right) \leqslant 2k - \lambda_i(L_G), \qquad i = 1, \ldots, n,$$

$$2k - \lambda_{m+n-i+1}(L_G) \leqslant 2 + 2\cos\left(\frac{2\pi\lfloor (2n-i+1)/2 \rfloor}{n}\right), \tag{1}$$

$$i = m+1, \ldots, 2n.$$

Since G is k-regular, $Q_G = 2kI_n - L_G$; hence, $2k - \lambda_i(L_G) = \lambda_{n-i+1}(Q_G)$. Also, $2 + 2\cos((2\pi\lfloor i/2 \rfloor)/n) = \lambda_{n-i+1}(Q_{C_n})$, so the first part of (1) is equivalent to

$$\lambda_{n-i+1}(Q_{C_n}) \leqslant \lambda_{n-i+1}(Q_G), \qquad i = 1, \ldots, n,$$

or

$$\lambda_i(Q_{C_n}) \leqslant \lambda_i(Q_G), \qquad i = 1, \ldots, n. \tag{2}$$

In the same way we see that $2k - \lambda_{m+n-i+1}(L_G) = \lambda_{i-m}(Q_G)$ and $2 + 2\cos((2\pi\lfloor (2n-i+1)/2 \rfloor)/n) = \lambda_{i-n}(Q_{C_n})$, so the second part of (1) becomes

$$\lambda_{i-m}(Q_G) \leqslant \lambda_{i-n}(Q_{C_n}), \qquad i = m+1, \ldots, 2n,$$

which is equivalent to

$$\lambda_{i-m+n}(Q_G) \leqslant \lambda_i(Q_{C_n}), \qquad i = m - n + 1, \dots, n. \tag{3}$$

The equations (2) and (3) follow directly from Theorem 1′. ∎

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