Hamilton Cycles in Claw-Free Graphs

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ABSTRACT

Bondy conjectured that if G is a k-connected graph of order n such that

$$\sum_{v \in I} d(v) \geq n + k(k-1)$$

for any $(k+1)$-independent set I of G, then the subgraph outside any longest cycle contains no path of length $k-1$. In this paper, we are going to prove that, if G is a k-connected claw-free ($K_{1,3}$-free) graph of order n such that

$$\sum_{v \in I} d(v) \geq n - k$$

for any $(k+1)$-independent set I, then G contains a Hamilton cycle. The theorem in this paper implies Bondy's conjecture in the case of claw-free graphs.

A graph is called "claw-free" if G has no induced $K_{1,3}$ subgraph. Mathews and Sumner showed the following theorem:

Theorem 1 [3]. If G is a claw-free 2-connected graph of order n with minimum degree δ such that $3\delta \geq n - 2$, then G contains a Hamilton cycle.

In this paper, we present a result about Hamilton cycles in k-connected claw-free graphs *which* generalizes the Mathews–Sumner result.

Theorem 2. Let G be a k-connected claw-free graph of order n such that

$$k \geq 2 \quad \text{and} \quad \sum_{v \in I} d(v) \geq n - k\,,$$

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for any $(k + 1)$-independent set I. Then G contains a Hamilton cycle.

Figure 1 shows an example in which the existence of the Hamilton cycle can be concluded by Theorem 2 but not by Theorem 1. Bondy has a well-known conjecture about the longest cycle and the degree sum condition.

Conjecture [1]. If G is a k-connected graph of order n such that

$$\sum_{v \in I} d(v) \geq n + k(k - 1)$$

for any $(k + 1)$-independent set I of G, then the subgraph outside any longest cycle contains no path of length $k - 1$.

Theorem 2 implies the conjecture in the case of claw-free graphs.

In this paper, let

$$N_D(v) = \{u \in V(D) \mid (v, u) \in E(G)\},$$

where D is a subgraph of G. If $V(D) = V(G)$, we simply write $N(v)$ instead of $N_D(v)$. If $C = x_1 \cdots x_r x_1$ is a cycle, $x_i C x_j$ denotes the interval $x_i x_{i+1} \cdots x_{j-1} x_j$ of C and $x_j \overline{C} x_i$ denotes the interval $x_j x_{j-1} \cdots x_{i+1} x_i$ of C.

PROOF OF THEOREM 2.

Let $G = (V, E)$ be a graph satisfying the conditions given in the theorem. Let $C = v_1 \cdots v_r v_1$ be a longest cycle of G. Assume that C is not a Hamilton cycle. Let B be a component of $G \backslash V(C)$.

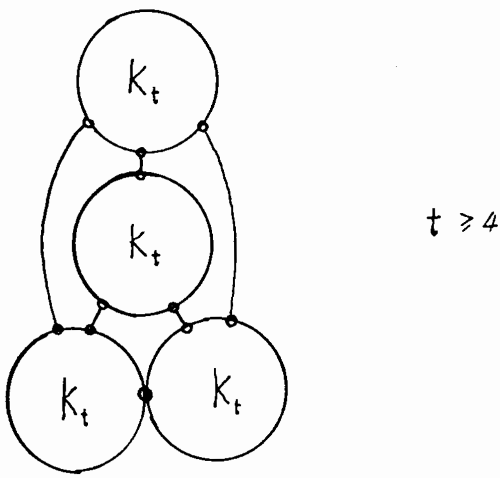

FIGURE 1.

By k-connectivity, there are h edges joining B and C, $h \geq k \geq 2$. Notice that $h \geq |N_C(x)|$ for any $x \in V(B)$. Let these edges be $\{(x_i, v_{j_i}) \mid i = 1, \ldots, h\}$, where $x_i \in V(B)$ and $v_{j_i} \in V(C)$, for $i = 1, \ldots, h$ and $1 \leq j_1 < j_2 < \cdots < j_h \leq r$. Let $x_i B x_j$ denote a path of B joining x_i and x_j.

We define some special sets on C by use of the following algorithm.

Algorithm (v_{j_μ}).

(a) $w_1 \cdots w_r \leftarrow v_{j_\mu - 1}\overline{C}v_{j_\mu}$ and $S_\mu \leftarrow \emptyset$.

(b) If there is an integer i such that $w_i, w_{i+1} \in N(w_1)$, choose i as big as possible. The pair $\{w_i, w_{i+1}\}$ is called the *insertion pair* for w_1.
$S_\mu \leftarrow S_\mu \cup \{w_1\}$,
$w_1 \cdots w_r \leftarrow w_2 \cdots w_i w_1 w_{i+1} \cdots w_r$;

(c) Repeat (b) until either $w_{i \pm 1} \notin N(w_1)$ for all $w_i \in N(w_1)$ or $w_1 \in S_\mu$; If $w_1 \notin S_\mu$, then $S_\mu \leftarrow S_\mu \cup \{w_1\}$;

(d) Stop.

Using Algorithm (v_{j_μ}) for each $\mu = 1, \ldots, h$, we obtain sets $S_1, \ldots, S_h$. Let

$$\bigcup_{v \in S_i} N_C(v) = N_i .$$

Proposition 1. We have

$$S_1 \subseteq \{v_{j_h+1}, \ldots, v_{j_1-1}\},$$
$$S_t \subseteq \{v_{j_{t-1}+1}, \ldots, v_{j_t-1}\} \qquad \text{for } t = 2, \ldots, h,$$

and

$$S_i \cap S_j = \emptyset \qquad \text{if} \quad i \neq j .$$

Proof. From Algorithm (v_{j_μ}) it is obvious that S_μ is an interval on C for $\mu = 1, \ldots, h$. Now $v_{j_h} \notin S_1$, because otherwise, during the execution of Algorithm (v_{j_1}), we would have a path $w_1 \cdots w_r$ with $w_1 = v_{j_h}$ and $w_r = v_{j_1}$. Then the cycle $w_1 \cdots w_r x_1 B x_h w_1$ would be longer than C. Hence, $S_1 \subseteq \{v_{j_h+1}, \ldots, v_{j_1-1}\}$ and the other containments can be determined in a similar manner. ∎

Consider $S_1 = \{v_{j_1-1}, \ldots, v_{j_1-|S_1|}\}$. There exists the least integer α_1 such that the insertion pair for $v_{j_1-\alpha_1}$ is not contained in $C \backslash [S_1 \backslash \{v_{j_1-1}, \ldots, v_{j_1-\alpha_1}\}]$. When Algorithm ($v_{j_1}$) stops, either w_1 has no insertion pair or $w_1 \in S_1$ (that is, $w_1 = v_{j_1-\alpha}$ has its intersection pair intersecting with $\{v_{j_1-\alpha-1}, \ldots, v_{j_1-|S_1|}\}$). Both cases guarantee the existence of $v_{j_1-\alpha_1}$. If $\alpha_1 < |S_1|$, then $v_{j_1-\alpha_1}$ is the first vertex whose insertion pair intersects $\{v_{j_1-\alpha_1-1}, \ldots, v_{j_1-|S_1|}\}$ during the execution of Algorithm (v_{j_1}). Hence, *the insertion pair for* $v_{j_1-\alpha_1}$ *is contained in* $\{v_{j_1-\alpha_1-1}, \ldots, v_{j_1-|S_1|-1}\}$ *when* $\alpha_1 < |S_1|$. *And* $v_{j_1-\alpha_1}$ *has no insertion pair when* $\alpha_1 = |S_1|$. Similarly, consider $\mu = 1, \ldots, h$; we will obtain $v_{j_\mu-\alpha_\mu}$.

The remainder of this proof will show that

(a) $I = \{x, v_{j_1-\alpha_1}, \ldots, v_{j_h-\alpha_h}\}$ is an independent set, where $x \in B$;
(b) $B, N(v_{j_1-\alpha_1}), \ldots, N(v_{j_h-\alpha_h}), \{v_{j_1}, \ldots, v_{j_h}\}$ and $\{v_{j_1-\alpha_1}, \ldots, v_{j_h-\alpha_h}\}$, are disjoint sets.

If we can verify Assertions (a) and (b), then I *will be an independent set contradicting the conditions of the theorem.*

During the execution of Algorithm (v_{j_μ}), we produce an operation on some paths $P = w_1 \cdots w_p$. Assume only one of $\{w_1, w_p\}$ is in S_μ (say, w_1) and let $\{w_i, w_{i+1}\}$ be the insertion pair for w_1. Define

$$Z_\mu(P) = w_2 \cdots w_i w_1 w_{i+1} \cdots w_p\,.$$

(If $w_p \in S_\mu$, then $Z_\mu(P) = w_1 \cdots w_i w_p w_{i+1} \cdots w_{p-1}$.) Hence, we can define the operation Z_μ on some paths $P = w_1 \cdots w_p$ with respect to S_μ. *The path* $Z_\mu(P)$ *is well defined only when* $|\{w_1, w_p\} \cap S_\mu| = 1$ *and the insertion pair for the vertex in this intersection exists. When* Z_μ *is operating on* P, *the endvertex* w_1 *(or* w_p*) will be moved between its insertion pair.* And $Z_\mu^\beta(P)$ denotes the compositions of the operation Z_μ on P (repeated β times).

We claim that

$$(v_{j_\mu}, v_{j_\mu-\alpha_\mu}) \notin E(G) \tag{1}$$

and

$$\alpha_\mu > 1 \qquad \text{for} \quad \mu = 1, \ldots, h\,.$$

Without loss of generality, consider $\mu = 1$. Let $P = v_{j_1-1}\overline{C}v_{j_1}$ and $w_1 \cdots w_r = Z_1^{\alpha_1-1}(P)$ where $w_1 = v_{j_1-\alpha_1}, w_r = v_{j_1}$. Assume that $(w_1, w_r) \in E(G)$. Then $w_1 \cdots w_r w_1$ is also a longest cycle. Now $\{x_1, w_1, w_{r-1}\} \subseteq N(w_r)$ and G being claw-free imply that $(w_1, w_{r-1}) \in E(G)$. So (w_{r-1}, w_r) is the insertion pair for w_1, which contradicts that the insertion pair for w_1 is not contained in $V(C)\backslash\{v_{j_1-\alpha_1-1}, \ldots, v_{j_1-|S_1|}\}$. Hence, we must have that $(w_1, w_r) = (v_{j_1}, v_{j_1-\alpha_1}) \notin E(G)$. Incidentally, the same proof shows that $\alpha_1 > 1$ always holds.

Let us consider $v_{j_1-\alpha_1}$ and $v_{j_\lambda-\alpha_\lambda}$ as an example. Let $P = v_{j_1-1}\overline{C}v_{j_1}$ and $Q = v_{j_1-1}\overline{C}v_{j_\lambda}x_\lambda B x_1 v_{j_1} C v_{j_\lambda-1}$ (let $q = |Q|$).

Proposition 2. Let $\alpha < |S_1|$. We will show the following:

(a) $Z_1^\alpha(P)$ is well defined; let $Z_1^\alpha(P) = w_1 \cdots w_r$.
(b) $v_{j_\lambda-1}, v_{j_\lambda}$ are adjacent in $Z_1^\alpha(P)$ and $v_{j_\lambda-1} \notin N(w_1)$.
(c) $Z_1^\alpha(Q)$ is well defined.
(d) Let $z, z' \in N(w_1)$. Then z and z' are adjacent in $Z_1^\alpha(P)$ if and only if z and z' are adjacent in $Z_1^\alpha(Q)$.

Proof. Clearly, (a) is true for all α, $0 \le \alpha < |S_1|$. We prove (b) by induction on α. If $(v_{j_\lambda - 1}, v_{j_1 - 1}) \in E(G)$, the cycle $v_{j_1-1}\overleftarrow{C}v_{j_\lambda}x_\lambda B x_1 v_{j_1} C v_{j_\lambda - 1} C v_{j_1 - 1}$ would be longer than C. So it is true for $\alpha = 0$. Assume that it is true for $\alpha < \kappa$. Since $v_{j_\lambda - 1}, v_{j_\lambda}$ are adjacent in $Z_1^{\kappa-1}(P)$, let $Z_1^{\kappa-1}(P) = u_1 \cdots u_r$ with $v_{j_\lambda} = u_i$, $v_{j_\lambda - 1} = u_{i+1}$. Since $v_{j_\lambda - 1} \notin N(u_1)$, the insertion pair for u_1 will not be $\{v_{j_\lambda - 1}, v_{j_\lambda}\} = \{u_1, u_{i+1}\}$. Hence, $v_{j_\lambda - 1}, v_{j_\lambda}$ are still adjacent in $Z_1^\kappa(P)$. If $v_{j_\lambda - 1} \in N(w_1)$, then the cycle $w_1 Z_1^\kappa(P) v_{j_\lambda} x_\lambda B x_1 w_r \overline{Z_1^\kappa(P)} v_{j_\lambda - 1} w_1$ would be longer than C.

By (b), the insertion pair for w_1 is always contained in either $\{w_1, \ldots, w_i\}$ or $\{w_{i+1}, \ldots, w_r\}$, where $v_{j_\lambda - 1} = w_{i+1}$ and $v_{j_\lambda} = w_i$. Hence,

$$Z_1^\alpha(Q) = w_1 Z_1^\alpha(P) w_i x_\lambda B x_1 w_r Z_1^\alpha(\overline{P}) w_{i+1}$$

and $Z_1^\alpha(Q)$ is well defined. Let $z, z' \in N(w_1)$. If z, z' are adjacent in $Z_1^\alpha(P)$ [respectively, in $Z_1^\alpha(Q)$], then either $z, z' \in \{w_1, \ldots, w_i\}$ or $z, z' \in \{w_{i+2}, \ldots, w_r\}$. Hence, z, z' are adjacent in $Z_1^\alpha(Q)$ [respectively, in $Z_1^\alpha(P)$). ∎

Proposition 3.

(a) If $\alpha < |S_1|$ and $\beta < |S_\lambda|$, then $Z_1^\alpha Z_\lambda^\beta(Q)$ is well defined.

(b) Let $\{Z_{\mu_1}, \ldots, Z_{\mu_s}\}$ be a series of operations, where $\mu_1, \ldots, \mu_s \in \{1, \lambda\}$. $Z_{\mu_s} Z_{\mu_{s-1}} \cdots Z_{\mu_1}(Q)$ is only dependent on the number of Z_1 and the number of Z_λ. In other words, any permutation of $\{\mu_1, \ldots, \mu_s\}$ would not make any difference in $Z_{\mu_s} \cdots Z_{\mu_1}(Q)$.

(c) Let $Z_1^\alpha(P) = w_1 \cdots w_r$, $Z_1^\alpha Z_\lambda^\beta(Q) = u_1 \cdots u_q$, and $v, v' \in N(w_1)$. Then v, v' are adjacent in $Z_1^\alpha(P)$ if and only if v, v' are adjacent in $Z_1^\alpha Z_\lambda^\beta(Q)$.

Proof. We use induction on $\alpha + \beta$. When $\beta = 0$, (a) and (c) are true by Proposition 2, and (b) is true because $\beta = 0$. Similarly, the proposition is true when $\alpha = 0$.

Assume that (a) and (b) are true for $\alpha + \beta < \kappa$ ($\kappa \ge 2$). Consider that $\alpha + \beta = \kappa$, $\alpha < |S_1|$, and $\beta < |S_\lambda|$. We only need to show that $Z_1^\alpha Z_\lambda^\beta(Q)$ is well defined and

$$Z_1 Z_\lambda Z_1^{\alpha-1} Z_\lambda^{\beta-1}(Q) = Z_\lambda Z_1 Z_1^{\alpha-1} Z_\lambda^{\beta-1}(Q).$$

By the induction hypothesis, $Z_1^{\alpha-1} Z_\lambda^{\beta-1}(Q)$, $Z_\lambda Z_1^{\alpha-1} Z_\lambda^{\beta-1}(Q)$, and $Z_1^\alpha Z_\lambda^{\beta-1}(Q)$ are well defined. Let $Z_1^{\alpha-1} Z_\lambda^{\beta-1}(Q) = w_1 \cdots w_q = Q^*$, $w_1 \in S_1$, and $w_q \in S_\lambda$. Let $\{w_a, w_{a+1}\}$ be the insertion pair for w_1, and let $\{w_b, w_{b+1}\}$ be the insertion pair for w_q—all of which exist because $Z_\lambda(Q^*)$ and $Z_1(Q^*)$ are well defined. If $\{w_a, w_{a+1}\} = \{w_b, w_{b+1}\}$, we would get a cycle $w_1 w_{a+1} Q^* w_q w_a \overline{Q}^* w_1$ longer than C. Hence, $\{w_a, w_{a+1}\} \ne \{w_b, w_{b+1}\}$ and $Z_1 Z_\lambda(Q^*) = Z_\lambda Z_1(Q^*) = w_2 \cdots w_a w_1 w_{a+1} \cdots w_b w_q w_{b+1} \cdots w_{q-1}$ when $a < b$ or $Z_1 Z_\lambda(Q^*) = Z_\lambda Z_1(Q^*) = w_2 \cdots w_b w_q w_{b+1} \cdots w_a w_1 w_{a+1} \cdots w_{q-1}$ when $b < a$ and therefore (a) and (b) follow.

Assume that (c) is true for $\alpha + \beta < \kappa$ ($\kappa \geq 2$) and consider $\alpha + \beta = \kappa$. Let $v, v' \in N(v_{j_1-\alpha-1})$ and by the induction hypothesis, v, v' are adjacent in $Z_1^\alpha(P)$ if and only if v, v' are adjacent in $Z_1^\alpha Z_\lambda^{\beta-1}(Q) = y_1 \cdots y_q = Q^{**}$ (where $y_1 = v_{j_1-\alpha-1}$). Suppose that v, v' are adjacent in $Z_1^\alpha Z_\lambda^{\beta-1}(Q)$. The insertion pair for y_q in Q^{**} is not $\{v, v'\}$. If so, let $\{v, v'\} = \{y_i, y_{i+1}\}$ and then the cycle $y_1 Q^{**} y_i y_q \overline{Q}^{**} y_{i+1} y_1$ would be longer than C. Hence, v, v' are still adjacent in $Z_1^\alpha Z_\lambda^\beta(Q)$. Conversely, suppose that v, v' are adjacent in $Z_1^\alpha Z_\lambda^\beta(Q)$, but not in $Z_1^\alpha Z_\lambda^{\beta-1}(Q) = Q^{**}$. Then, $y_q \in \{v, v'\} \subseteq N(y_1)$, which would give a cycle $y_1 \cdots y_q y_1$ longer than C. So v, v' must be adjacent in $Z_1^\alpha Z_\lambda^{\beta-1}(Q)$. ∎

We claim that

$$B \cap N(v_{j_1-\alpha_1}) = \emptyset \tag{2}$$

and

$$(v_{j_1-\alpha_1}, v_{j_\lambda-\alpha_\lambda}) \notin E(G). \tag{3}$$

If $x \in B \cap N(v_{j_1-\alpha_1})$, then the cycle $Z_1^{\alpha_1-1}(P) v_{j_1} x_1 B x v_{j_1-\alpha_1}$ would be longer than C. If $(v_{j_1-\alpha_1}, v_{j_\lambda-\alpha_\lambda}) \in E(G)$, then the cycle $Z_1^{\alpha_1-1} Z_\lambda^{\alpha_\lambda-1}(Q) v_{j_\lambda-\alpha_\lambda} v_{j_1-\alpha_1}$ would be longer than C. Hence, we have proved the first assertion.

We claim that

$$N_1 \cap S_\lambda = \emptyset \quad \text{and} \quad N_\lambda \cap S_1 = \emptyset. \tag{4}$$

If not, let $v_{j_1-\alpha-1} \in N(v_{j_\lambda-\beta-1}) \cap S_1 \subseteq N_\lambda \cap S_1$, where $\alpha < |S_1|$ and $\beta < |S_\lambda|$. Then the cycle $Z_1^\alpha Z_\lambda^\beta(Q) v_{j_\lambda-\beta-1} v_{j_1-\alpha-1}$ would be longer than C. Hence, $v_{j_1-\alpha_1} \notin N_\lambda$ and $v_{j_\lambda-\alpha_\lambda} \notin N_1$.

Let $Z_1^\alpha Z_\lambda^\beta(Q) = w_1 \cdots w_q$ ($\alpha \leq \alpha_1 - 1, \beta \leq \alpha_\lambda - 1$). Then we claim that $\{v_{j_1-\alpha-1}, v_{j_1-\alpha-2}, \ldots, v_{j_1-|S_1|-1}\}$ remains as an interval in $Z_1^\alpha Z_\lambda^\beta(Q)$ and $w_1 = v_{j_1-\alpha-1}, \ldots, w_{|S_1|-\alpha+1} = v_{j_1-|S_1|-1}$. By the choice of α_1, it is obviously true when $\beta = 0$. We proceed by induction on β. Let $Q^* = Z_1^\alpha Z_\lambda^{\beta-1}(Q) = y_1 \cdots y_q$. Since $N_\lambda \cap S_1 = \emptyset$, by (4), the insertion pair for y_q will not be contained in $\{y_1, \ldots, y_{|S_1|-\alpha+1}\}$. Hence,

$$\{y_1, \ldots, y_{|S_1|-\alpha+1}\} = \{w_1, \ldots, w_{|S_1|-\alpha+1}\}$$

remains as an interval in $Z_1^\alpha Z_\lambda^\beta(Q)$.

We claim that

$$N(v_{j_1-\alpha_1}) \cap N(v_{j_\lambda-\alpha_\lambda}) = \emptyset, \tag{5}$$

which is a part of the second assertion. Let $Q^* = Z_1^{\alpha_1-1} Z_\lambda^{\alpha_\lambda-1}(Q) = w_1 \cdots w_q$. By (2), $N(v_{j_1-\alpha_1}) \cap N(v_{j_\lambda-\alpha_\lambda}) \cap V(B) = \emptyset$. If $y \in N(v_{j_1-\alpha_1}) \cap N(vj_\lambda - \alpha_\lambda) \subseteq [V(G) \backslash (C \cup B)]$, then the cycle $Z_1^{\alpha_1-1} Z_\lambda^{\alpha_\lambda-1}(Q) v_{j_\lambda-\alpha_\lambda} y v_{j_1-\alpha_1}$ would be longer than C. Assume that $w_s \in N(v_{j_1-\alpha_1}) \cap N(v_{j_\lambda-\alpha_\lambda}) \cap V(C)$. If $(w_1, w_q) \in E(G)$, then the cycle $w_1 \cdots w_q w_1$ would be longer than C. If $(w_q, w_{s-1}) \in E(G)$ or $(w_1, w_{s+1}) \in E(G)$, the cycle $w_1 Q^* w_{s-1} w_q \overline{Q}^* w_s w_1$ or $w_1 Q^* w_s w_q \overline{Q}^* w_{s+1} w_1$ would be longer than C. If $(w_{s-1}, w_{s+1}) \in E(G)$, the cycle $w_1 Q^* w_{s-1} w_{s+1} Q^* w_q w_s w_1$ would be longer than C. Since G is a claw-free graph, the only remaining cases are that $(w_1, w_{s-1}) \in E(G)$ when $s - 1 > 1$ and $(w_q, w_{s+1}) \in E(G)$ when $s + 1 < q$. Note that $s - 1 = 1$ and $s + 1 = q$ cannot hold simultaneously, as this would imply $3 = q \geq |C| + 1$. If $s - 1 > 1$ by Proposition 3, then the adjacent pair $w_{s-1}, w_s \in N(w_1)$ implies the existence of the insertion pair for w_1 in $Z_1^{\alpha_1 - 1}(P)$. By the choice of α_1, we have that

$$\{w_{s-1}, w_s\} \subseteq \{v_{j_1-\alpha_1-1}, \ldots, v_{j_1-|S_1|-1}\}.$$

Similarly, if $s + 1 < q$, then

$$\{w_{s+1}, w_s\} \subseteq \{v_{j_\lambda-\alpha_\lambda-1}, \ldots, v_{j_\lambda-|S_\lambda|-1}\}.$$

If $s - 1 > 1$ and $s + 1 < q$, then

$$w_s \in \{v_{j_1-\alpha_1-1}, \ldots, v_{j_1-|S_1|-1}\} \cap \{v_{j_\lambda-\alpha_\lambda-1}, \ldots, v_{j_\lambda-|S_\lambda|-1}\},$$

which contradicts Proposition 1. So without loss of generality, let $s - 1 > 1$ and $s + 1 = q$. Since $\{w_1, \ldots, w_s\}$ is a subset of $\{v_{j_1-\alpha_1}, \ldots, v_{j_1-|S_1|-1}\}$ and $\{v_{j_1-\alpha_1}, \ldots, v_{j_1-|S_1|-1}\}$ remains as an interval in $Z_1^{\alpha_1-1} Z_\lambda^{\alpha_\lambda-1}(Q)$, we must have that $2 \geq |Q \backslash \{v_{j_1-\alpha_1}, \ldots, v_{j_1-|S_1|}\}| \geq |Q \backslash S_1|$. It contradicts that $|Q \backslash S_1| \geq |S_\lambda| + |B| + |\{v_{j_\lambda}\}| \geq 3$.

We now wish to show that

$$(v_{j_\lambda}, v_{j_1-\alpha_1}) \notin E(G). \tag{6}$$

Since $\{v_{j_\lambda-1}, v_{j_\lambda+1}, x_\lambda\} \subseteq N(v_{j_\lambda})$, G is a claw-free graph, and C is a longest cycle, note that $(v_{j_\lambda-1}, v_{j_\lambda+1}) \in E(G)$. Suppose that $(v_{j_\lambda}, v_{j_1-\alpha_1}) \in E(G)$, let α be the least integer such that $(v_{j_\lambda}, v_{j_1-\alpha_1}) \in E(G)$. Then the insertion pair for $v_{j_1-\gamma}$ is not $\{v_{j_\lambda}, v_{j_\lambda+1}\}$ for $\gamma = 1, \ldots, \alpha - 1$. Hence, v_{j_λ} and $v_{j_\lambda+1}$ are adjacent in $Z_1^{\alpha-1}(P) = P^* = w_1 \cdots w_r$ where $w_1 = j_{1-\alpha}$ and $w_r = v_{j_1}$. Also, v_{j_λ} and $v_{j_\lambda-1}$ are adjacent in P^* by Proposition 2. Let $w_{i-1} = v_{j_\lambda+1}$, $w_i = v_{j_\lambda}$, $w_{i+1} = v_{j_\lambda-1}$. Then the cycle $w_1 P^* w_{i-1} w_{i+1} P^* w_r x_1 B x_\lambda w_i w_1$ would be longer than C. Therefore we conclude that $v_{j_\lambda} \notin N(v_{j_1-\alpha_1})$.

All results we have obtained hold if $\{1, \lambda\}$ is replaced by any pair $\{s, t\} \subseteq \{1, \ldots, h\}$.

Now we can establish the second assertion:
$N(v_{j_s-\alpha_s}) \cap N(v_{j_t-\alpha_t}) = \emptyset$ (by 5), $N(v_{j_s-\alpha_s}) \cap \{v_{j_1}, \ldots, v_{j_h}\} = \emptyset$ (by 1 and 6), $N(v_{j_s-\alpha_s}) \cap \{v_{j_1-\alpha_1}, \ldots, v_{j_h-\alpha_h}\} = \emptyset$ (by 3), $\{v_{j_1-\alpha_1}, \ldots, v_{j_h-\alpha_h}\} \cap \{v_{j_1}, \ldots, v_{j_h}\} = \emptyset$ (by Proposition 1), and $V(B) \cap (v_{j_s-\alpha_s}) = \emptyset$ (by 2).

Hence, $V(B)$, $N(v_{j_1-\alpha_1}), \ldots, N(v_{j_h-\alpha_h})$, $\{v_{j_1}, \ldots, v_{j_h}\}$, and $\{v_{j_1-\alpha_1}, \ldots, v_{j_h-\alpha_h}\}$ are disjoint sets of $V(G)$. If we let $I' = \{v_{j_1-\alpha_1}, \ldots, v_{j_h-\alpha_h}\}$, then

$$\sum_{v \in I'} d(v) \leq |V(G) \backslash V(B)| - 2h .$$

Let $I = I' \cup \{x\}$ for any $x \in V(B)$. Recall that $|N_C(x)| \leq h$ by the definition of h. Then

$$\begin{aligned} \sum_{v \in I} d(v) &\leq \sum_{v \in I'} d(v) + (|V(B) \backslash \{x\}| + h) \\ &\leq n - h - 1 \\ &\leq n - k - 1 . \end{aligned}$$

Therefore any $(k + 1)$-subset of I will have degree sum less than $n - k$, which contradicts the condition of the theorem. We conclude that G has a Hamilton cycle.

Remark. Recently, Theorem 2 in the case of $k = 2$ was independently proved by H. J. Broersma [2].

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