Arithmetic Explorations 2

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December 26, 2022

This Part 2 of my Arithmetic Explorations - Click here to see part 1.

1 Swaps of Numerical Semigroups

Let's start this one with a graph! We've discussed the semigroup tree by adding the Frobenius number, decreasing the genus by 1. And we've discussed the ordinarization number, which took the form of adding the Frobenius number and removing the Multiplicity, keeping the genus the same. Repeating this always stabilizes at the ordinary subgroup $\mathcal{O}_g = \{1, 2, \ldots, g\}^c$.

We discussed inverting these operations. To undo the addition of a Frobenius number, we must look for a generator that is larger than the Frobenius number, which we call an *effective generator*. To undo the removal of the Multiplicity, we need to look for a gap smaller than the multiplicity that is *fragile*, which we call a *removable gap*.

These came from the broader fact that

$$S \cup \{x\} \ \textit{numerical semigroup} \Leftrightarrow x \ \textit{fragile}$$

$$S - \{x\} \ \textit{numerical semigroup} \Leftrightarrow x \ \textit{irreducible}$$

The effective generator bit and removable gap bit are just added to specifically undo the addition of Frobenius elements, or removal of multiplicities.

But what if we simply look for any pair $(x, y) \in S^c \times S$ and see if

$$S^{xy} := S \cup \{x\} - \{y\}$$

is a numerical semigroup? The ordinarization sequence came from seeing this is true for $x = F(S)$ and $y = m(S)$. A graph is a great way to visualize this. Let's call the graph formed from all such pairs $\mathcal{S}_g$.

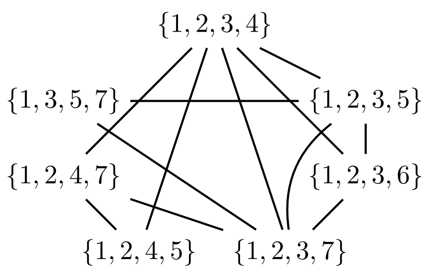


Figure 1: $\mathcal{S}_4$

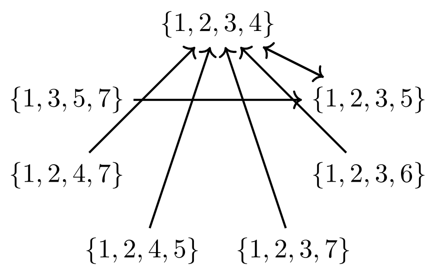


Figure 2: $x = F(S), y = m(S)$

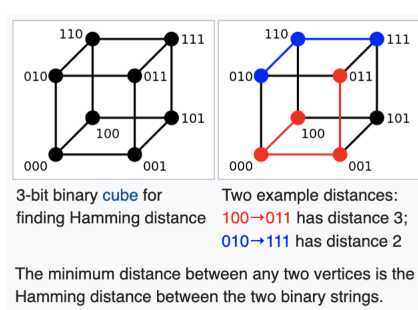
With the graph as a visual, we see another definition that we can use:

$$(S_1, S_2) \in E(\mathcal{S}_g) \iff |S_1 \triangle S_2| = 2$$

Those two elements are the ones we're able to swap out and get another numerical semigroup. I like this point of view because (1) we've seen that the symmetric difference of graphs (and specifically matchings) was important for the theorem showing ***a matching M is maximal if and only if there is no augmenting path in G with M.*.

And (2), this notion is an instance of a much more general idea called The Hamming Distance. Usually defined in terms of strings, the *Hamming Distance* between two strings of equal lengths is the the number of spots where they disagree. If we think of the strings as sets S_1 and S_2 of equal length, then the set $S_1 \triangle S_2$ is exactly the set of elements where they disagree, hence $(1/2)|S_1 \triangle S_2|$ will be their Hamming distance. This notion of distance is very helpful in error-correcting codes.

Which means given two numerical semigroups S, T of genus g, they are connected in $\mathcal{S}_g$ if and only if $d_H(S^c, T^c) = 1$. Graphs like this are called Hamming Graphs. Though that term specifically corresponds to taking *all* d-element subsets of some finite set S of size q and connecting two of them if their hamming distance is 1, denoted by $H(d, q)$. Lot's of cool graphs can be formed in this way: for example, the cube!



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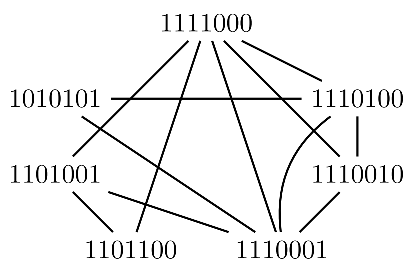
A lot of other graphs are defined in similar ways: The Kneser Graphs, The Johnson Graphs, Generalized Kneser graphs - all in terms of intersections. What we want is a paper defined by the symmetric difference of such sets. I also can't help but mention here a really cool paper that takes a graph G and studies the Hamming distances of the vectors of its adjacency matrix, in terms of properties of the graph.

As explored in this paper Partial Hamming Graphs and Expansion Procedures by Bresar, we can define the notion of a partial Hamming graph by first defining an isometric subgraph. G_1 is an *isometric subgraph* of G_2 if the distance between any two vertices in G_1 is the same as their distance in G_2. That is, we don't remove any vertices that were instrumental in a minimal path between two vertices.

A *partial Hamming graph* is any graph whose vertices can be labeled with strings of length k so that the distance of any two vertices is equal to the Hamming distance of their strings. Clearly any isometric subgraph of $H(d, q)$ is a partial Hamming graph. Characterizing partial Hamming graphs is the goal of the above paper.

The strings assigned are obvious: binary strings of length $2g-1$ corresponding to the indicator function of S^c. For example, here is $\mathcal{S}_4$ with this viewpoint.



But as you can see, the hamming distance between connected strings is 2, instead of 1. Including the intermediate strings is akin to including the numerical semigroups of genus $g-1$ and $g+1$ as well. For example, $d_H(1111000, 1110001) = 2$ which corresponds to swapping 4 and 7 to get $\{1, 2, 3, 4\} \longleftrightarrow \{1, 2, 3, 7\}$. The intermediate semigroups we get corresponding to 1110000 and 1111001 are $\{1, 2, 3\}$ and $\{1, 2, 3, 4, 7\}$ respectively.

Anyway,

Conjecture 1 *The graph $\mathcal{S}_g$ is a Partial Hamming Graph*

A foundational paper by Chepoi in 1988 defines a type of *expansion* and shows that a graph G is a partial Hamming graph is and only if it can be obtained by a 1-vertex graph by a series of isometric expansions.

This is very cool because *if the conjecture is true*, then we could build $\mathcal{S}_g$ by a series of expansions from $\mathcal{O}_g$, which will give us a way to get all numerical semigroups of genus g. Analyzing the steps in the expansions could show $N(g) > N(g-1)$ for all g.

Let's try to detail some properties about $\mathcal{S}_g$. The number of vertices is clearly $N(g)$. The number of edges...maybe we compute via handshaking theorem? The degree of a vertex S is the number of pairs $(x, y) \in S^c \times S$ so that $S \cup \{x\} - \{y\}$ is a numerical semigroup. This means

$$y \text{ is irreducible in } S \cup \{x\} \Longrightarrow \forall s_1, s_2 \in S \cup \{x\},\ s_1 + s_2 \neq y$$

$$\Longrightarrow \forall s_1, s_2 \in S,\ s_1 + s_2 \neq y \Longrightarrow y \text{ is irreducible in } S$$

This also implies

$$\forall s \in S,\ s + x \neq y$$

The other direction is

$$x \text{ is fragile in } S - \{y\} \Longrightarrow \forall s \in S - \{y\},\ s + x \in S - \{y\}$$

$$\Longrightarrow \forall s \in S,\ s + x \in S \Longrightarrow x \text{ fragile in } S$$

It also implies

$$\forall s \in S,\ s + x \neq y$$

This is great, because I was worried that maybe we could swap two x and y that weren't fragile or irreducible separately. But this shows that if the swap is a numerical semigroup, then the elements are fragile/irreducible in the intermediate semigroups AND they are fragile/irreducible in the original semigroup.

As such, maybe it's better to define $\mathcal{S}_g$ like Γ_g: to have numerical semigroups of genus g and $g-1$ connected if their symmetric difference has 1 element. But for now we'll keep it the same, because I want to explore a sort of manifold-inspired idea.

I'm going this direction because I've been reading about local-to-global principles and would love a theorem of the following sort:

Theorem 1 (Prototype) *Suppose Γ is an infinite layered graph. Then global properties A of Γ imply properties B of the sequence $n_1, n_2, \ldots$.*

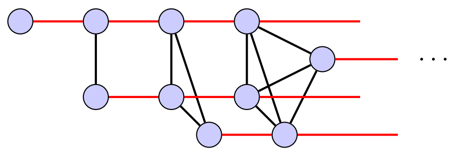
In particular, we want some properties that would force n_i to be increasing.

What's an infinite layered graph? I haven't found something like this but I'm sure I just need to look deeper into the established literature. It feels like an inverse limit of sorts. I define it as follows.

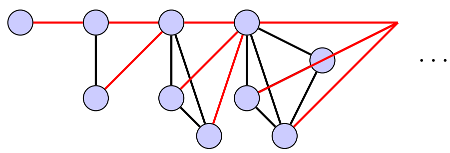
Definition 1 The graph Γ is called *infinitely layered* if it has a sequence of disjoint subgraphs $G_1, G_2, \ldots$ so that

1. Each $n_i = |V(G_i)|$ is finite
2. For all i, j, the set of edges $E(G_i, G_j)$ between G_i and G_j is empty if $|i - j| > 1$.
3. For each i, we have a transition map ψ_i telling us the connections between G_i and G_{i+1}.

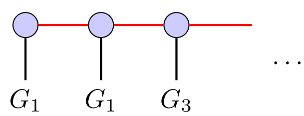
For example, if each $G_i = K_i$ is the complete graph on i vertices, and the transition map ψ_i is the identity map, then Γ would look like



While each map ψ_i could also be the constant map 1, which would look like



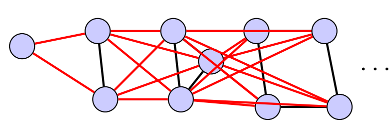
That's cool! It's like taking successive n-simplices and gluing them together at a single vertex. Any sequence of graphs $G_1, G_2, \ldots$ could be glued together intro an infinite layered graph via hanging them on a "clothesline":



This corresponds to choosing transition maps $\psi_i(1) = 1$ and $\psi_i(v) = 0$ otherwise (where 0 is interpreted as no edge).

Characteristics of Γ are closely related to the transition maps ψ_i. For example, taking the clothesline example, we see that most properties of Γ just come down to properties of the G_i. For example, coloring. If the chromatic number is uniformly bounded for all G_i, then the chromatic number of the clothesline is just the maximum of the chromatic numbers or the maximum plus 1 (if all 1st vertices are adjacent to edges of all colors).

If the transition maps are more complicated, we might increase the chromatic number more than 1, even if each individual graph has a uniform bound on chromatic number. For example, if all G_i are i-cycles but the transition maps are *complete maps*, meaning we connect every vertex in G_i to every vertex in G_{i+1}.



So we have the colorings of each individual graphs but must also consider the *transition colorings*, the colorings of the graphs formed by the vertices of G_i and G_{i+1} with edges given by ψ_i. The above graph can be described with layers C_i and transition graphs $K_{i,i+1}$, the complete bipartite graph on i and $i+1$ vertices.

Returning to the Partial Hamming Graph, I found a post by someone who has the exact same type of graph, asking what kind of Hamming graph it is. Here it is.

People gather a wonderful selection of ideas. For example, it led to the idea of Combinatorial Dichotomies, which seem very interesting! Otherwise, they basically address the fact that it is a subgraph of a Hamming Graph, and make a clear note that this *does not* mean it is a Partial Hamming Graph. Of course, we already saw that such a subgraph must be isometric to be a Partial Hamming Graph.

A further search reveals a paper by Sandi Klavzar and Iztok Peterin that characterizes subgraphs, induced subgraphs, and isometric subgraphs of Hamming graphs. After reading more about this, it seems doubtful that $\mathcal{S}_g$ would be an induced subgraph.

Let's go our own route and extend $\mathcal{S}_g$ to a general graph whose vertices are the g-element subsets of $\{1, 2, \ldots, 2g-1\}$ with a connection between two subsets if their symmetric difference has size 2. We'll call this $H_2(g)$. And I'll highlight the subgraph $\mathcal{S}_g$ in blue. Here are a couple of examples.

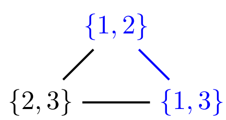


Figure 3: $H_2(2)$

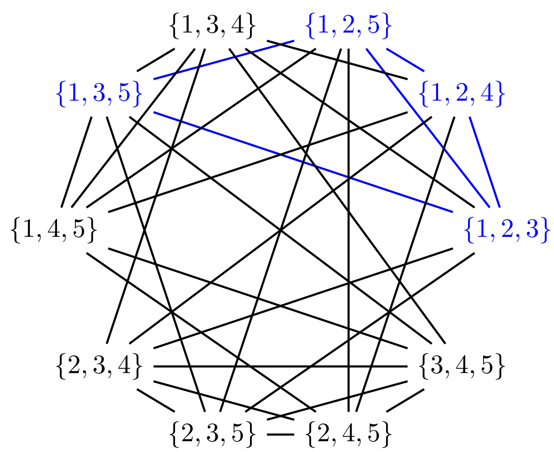


Figure 4: $H_2(3)$

It's clear that $\mathcal{S}_g$ is indeed an induced subgraph of $H_2(g)$. The properties of $H_2(g)$ are pretty easy to describe:

1. $H_2(g)$ has $\binom{2g-1}{g}$ vertices.

2. For a vertex v, each neighbor can be gotten by choosing an element in v and replacing it by an element in $[2g-1] \backslash v$. There are $g-1$ of these, and g elements in v, giving a total of $deg(v) = g(g-1)$

3. By the handshaking theorem, this tells us $H_2(g)$ has

$$\frac{1}{2}\binom{2g-1}{g}g(g-1) = \binom{2g-1}{g}\binom{g}{2}$$

edges.

4. In fact, I think $H_2(g)$ is a strongly-regular graph. Recall a $srg(v, k, \lambda, \mu)$ graph is one with v vertices, all degree k, and every pair of adjacent vertices have λ common neighbors, and every pair of non-adjacent vertices have μ common neighbors.

   If two sets S_1, S_2 are adjacent, then $S_1 \triangle S_2 = \{a, b\}$. If T is a common neighbor, then $T \triangle S_1 = \{t_1, s_1\}$ and $T \triangle S_2 = \{t_2, s_2\}$. If $s_1 = a$, then $t_2 = t_1$ and $s_2 = b$. In other words, we could choose any t_1 that is not in S_1 or S_2. We have

$$2g - 1 - |S_1 \cup S_2| = 2g - 1 - (g + 1) = g - 2$$

   .

   If $s_1 \neq a$, then $S_1 - \{s_1\}$ still contains a, so $T - \{t_1\}$ must also contain a. But S_2 does not contain a, so the only way for T and S_2 to be connected is if $t_2 = a$. Similarly, we'd need $t_1 = b$.

   So $T = S_1 \cup S_2 - \{s\}$ for some $s \neq a, b$ in S_1 or S_2. We have $g - 1$ choices for such an s. All together, this gives $2g - 3$ neighbors for any adjacent vertices.

5. If S_1 and S_2 are not adjacent, then $|S_1 \triangle S_2| > 2$. I'm afraid that the number of common neighbors might depend on the size of their symmetric difference. If T was a common neighbor, then $T \triangle S_1 = \{t_1, s_2\}$ and $T \triangle S_2 = \{t_2, s_2\}$. Then by associativity and commutativity of the symmetric difference,

$$(T \triangle S_1) \triangle (T \triangle S_2) = \{t_1, s_1, \} \triangle \{t_2, s_2\}$$

$$= (T \triangle T) \triangle (S_1 \triangle S_2) = S_1 \triangle S_2$$

   Since $|S_1 \triangle S_2| > 2$, we must have all t_i, s_i distinct and $S_1 \triangle S_2 = \{t_1, s_1, t_2, s_2\}$. So the only way two non-adjacent vertices have a common neighbor is if the distance between them is 2.

   This isn't so evident in the examples we did, but as g grows larger, we'll get sets that are further and further apart. We could consider this an *almost strongly regular graph* $asrg(v, k, \lambda_1, \lambda_2, \mu)$, where

$$v = |V|$$

$$deg(v) = k$$

$$\textit{All pairs } (u, v) \textit{ distance } 1 \textit{ apart}(i.e.edges) \textit{ have } \lambda_1 \textit{ common neighbors}$$

All pairs (u,v) distance 2 apart have λ_2 common neighbors

All pairs (u,v) distance ≥ 3 apart have μ common neighbors

> Now that I've written that, it actually makes sense as a generalization for strongly regular graphs. I'll look for places where this type of thing has been studied later.

Finishing up outside the list, if $S_1 \triangle S_2 = \{a_1, a_2, b_1, b_2\}$, then all common neighbors T are of the form $S_1 - \{a_i\} \cup \{b_j\}$, so we get $2 * 2 = 4$ possibilities.

So we've shown that these graphs $H_2(g)$ are

$$asrg\left(\binom{2g-1}{g},\ g(g-1),\ 2g-3,\ 4,\ 0\right)$$

This is great! Perhaps we can adapt the methods of counting induced subgraphs of strongly regular graphs (ex 1, and ex 2) to learn about the specific induced subgraph we're interested in!

For example, how many induced subgraphs on $N(g)$ vertices will be isomorphic to $\mathcal{S}_g$? Are there many that imitate this special graph related to numerical semigroups? Or does the special structure of numerical semigroups make this kind of subgraph rare?

Another way to look at the almost strongly regular graph is maybe as a *distance-regular graph*, where the number of neighbors between two vertices depends only on the distance between the two vertices. The following name is inspired by a type of function called a Radial Function if its value depends only on its distance from the origin.

Definition 2 A graph is a *radial graph* if there exists a function $\phi : \mathbb{Z}^+ \to \mathbb{N}$ so that all pairs of vertices u and v have $\phi(d(u,v))$ common neighbors. Note I'm using $\mathbb{N} = \{0, 1, 2, \dots\}$.

And literally just looking up Distance-Regular Graph shows a slightly stronger concept than what we want, so I'll stick with radial graphs!

Looking back at Bounds for regular induced subgraphs of strongly regular graphs by Evans, they give a wonderful exposition of the area. Naturally, the study of the spectrum of a graph (the eigenvalues of its adjacency matrix) are of deep importance here, because of Hoffman and Singleton of the Hoffman-Singleton Graph fame.

They characterized the possible eigenvalues of strongly regular graphs, as well as their multiplicities, in terms of the four parameters. This is immensely useful in so many areas, but it's near and dear to my heart (again) because I was introduced to it during my second summer research project!

We characterized the *critical group and smith group* of the *Rook's Graph R_n*. Remember from part 1 that the critical group of a graph is defined as the torsion part of the cokernel of the Laplacian. Which is a mouthful! But to break it down:

- A is the adjacency matrix of G
- D is the degree matrix of G

- The Laplacian $L = D - A$
- The cokernel $\mathbb{Z}^n/Im(L) \cong \mathbb{Z}^c \times \mathcal{K}(G)$, where c is the number of connected components of G and $\mathcal{K}(G)$ is *the critical group*.
- We define *the smith group* as the cokernel of the adjacency matrix itself.

Now the Rook's graph R_n is defined with it's vertices an $n \times n$ grid, with two vertices connected if and only if they lie in the same row or same column. In terms of chess, two vertices are adjacent if and only if you can get from one to the other with a *rook move*. You could similarly define other "chess graphs".

Using the fact that R_n is strongly regular, we can deduce the eigenvalues of it easily, and we can therefore figure out the size of $\mathcal{K}(G)$. To deduce the structure of the critical group, we used Norman Bigg's wonderful idea of a *chip-firing game* to encode the critical group. Let's describe that.

- A *configuration* on G is a a labeling of the vertices with some number of chips (even negative). We can think of this as a function $c : V \to \mathbb{Z}$.
- Addition of configurations is vertex-wise. These last two steps basically just identity G with $\mathbb{Z}^n$.
- The cokernel $\mathbb{Z}^n/Im(L)$ means our group is the set of configurations, but with a certain equivalence relation. A combinatorial interpretation of this relation is what Bigg's introduces.
- L always has the eigenvector $[1, 1, \ldots, 1]$ with eigenvalue 0, which gives the extra $\mathbb{Z}$ factor.
- The finite part $\mathcal{K}(G)$ can be thought of as the set of configurations that sum to zero, under the *chip-firing relation*:
  - *Fire vertex*: Subtract $deg(v)$ from v and add 1 to each neighbor of v.
  - *Pull vertex*: Add $deg(v)$ to v and subtract 1 from each neighbor.

  Recall the v^{th} column looks like $[0, -1, 0, 0, -1, deg(v), 0, -1, -1, 0]$. So we see that firing a vertex is just subtracting a column of L from a configuration.
- So two configurations are equivalent if you can fire and pull vertices to get from one to the other.

Fun Exercise: Try to prove this is finite using no linear algebra and only the combinatorial chip-firing game.

Pulling some nice graphics from our paper, here is an example of two equivalent configurations in $\mathcal{K}(G)$ by firing the red vertex and pulling the blue (note each box is a vertex).

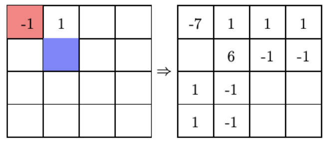


FIGURE 1. Equivalent configurations in $\mathcal{K}(R_4)$, seen by firing at the red vertex and pulling at the blue.

The fact that R_n was a grid made finding patterns in the critical group actually fairly accessible (with the help of our mentor Josh Ducey of course). Along with the automorphisms of R_n, which again were easy to visualize on the grid, we found 5 families of configurations that generate all of $\mathcal{K}(R_n)$!

$$c_1(i,j) = \begin{cases} -1, & \text{if } (i,j) = (1,1) \\ 1, & \text{if } (i,j) = (1,2) \\ 0, & \text{otherwise.} \end{cases}$$

$$c_2(i,j) = \begin{cases} -(n-1), & \text{if } (i,j) = (1,1) \\ 1, & \text{if } i = 1 \text{ and } 2 \le j \le n \\ 0, & \text{otherwise.} \end{cases}$$

$$c_3(i,j) = \begin{cases} -1, & \text{if } (i,j) = (1,1) \text{ or } (2,2) \\ 1, & \text{if } (i,j) = (1,2) \text{ or } (2,1) \\ 0, & \text{otherwise.} \end{cases}$$

$$c_4(i,j) = \begin{cases} 1, & \text{if } (i,j) = (1,1) \\ 0, & \text{otherwise.} \end{cases}$$

$$c_5(i,j) = \begin{cases} -n(n-1)(n-2), & \text{if } (i,j) = (1,1) \\ n(n-2), & \text{if } j = 1 \text{ and } 2 \le i \le n \\ 0, & \text{otherwise.} \end{cases}$$

We illustrate these five configurations in Figure 3.

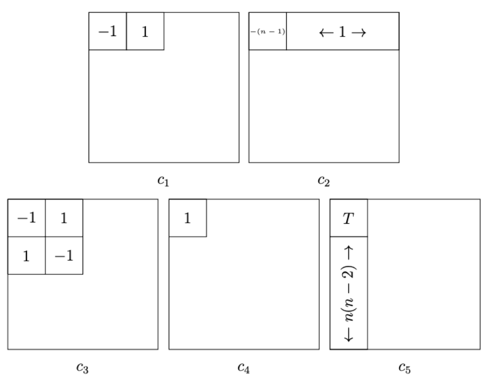


FIGURE 3. The main generators. Here $T = -n(n-1)(n-2)$.

With the important note that many of these came from eigenvectors! So with the Hoffman-Singleton theorem as a starting place, we were able to prove

$$\mathcal{K}(R_n) \cong (\mathbb{Z}/2n\mathbb{Z})^{(n-1)^2+1}(\mathbb{Z}/2n^2\mathbb{Z})^{2(n-2)}$$

Another place that strongly regular graphs take us is to Moore Graphs. These are graphs the maximize the number of vertices in a graph with diameter

k and girth $2k + 1$. A LOT has been studied about these graphs for many reasons, but one in particular is fascinating.

We know every Moore graph except for a single one, which must be a $srg(3250, 57, 0, 1)$. Its properties have been exstensively studied in various ways. For example, my mentor on the critical group project later published a paper characterizing $\mathcal{K}(G)$ of the missing Moore graph up to the 5-factor! Even though it may not exist, we have so much information about it. I find it very interesting.

Going full circle, the Ex 2 above on induced subgraphs of strongly regular graphs actually applies these to showing the automorphism group of the missing Moore graph must be uncharacteristically small.

2 Eigenvalues of Radial Graphs

Let's try to see if any of the work done on characterizing eigenvalues of strongly regular graphs $srg(v, k, \lambda, \mu)$ could be extended to our radial graphs $radial(v, k, \lambda(d))$, where $\lambda : \mathbb{Z}^+ \to \mathbb{N}$. Then the number of common neighbors of u and v is $\lambda(d(u, v))$. The two examples we've seen are

- A strongly regular graph has

$$\lambda(1) = \lambda, \quad \lambda(d) = \mu \; for \; d \; \geq 2$$

- The graph $H_2(g)$ has

$$\lambda(1) = 2g - 3, \; \lambda(2) = 4, \; \lambda(d) = 0 \; for \; d \; \geq 3$$

For any regular graph of degree k, we have $AJ = JA = kJ$, which is again recovering the all 1s eigenvalue. This is the same for radial graphs.

The next equation we get is in terms of 2-step paths:

$$A^2 = kI + \lambda A + \mu(J - I - A)$$

The (i, j) entry in A^2 is the number of paths of length 2 from i to j. The kI term comes from if $i = j$, going out and coming back along any edge adjacent to i. The λA term comes from when (i, j) is an edge. Then they have λ common neighbors, so we get a 2-step path from any of them. Hence λA.

If (i, j) is not an edge, then there are μ common neighbors, which accounts for the third term. The term $J - I - A$ looks complicated but it's just the matrix with a 1 everywhere there isn't an edge or along the diagonal. This is exactly what we want to multiply μ by.

For a radial graph $G = radial(v, k, \lambda)$, we would need to account for all distances pairs of vertices could take. The largest such distance is called the Diameter of G.

Two edges that are adjacent will share λ_1 common neighbors, so we retain the terms

$$A^2 = kI + \lambda_1 A + \dots$$

The last term needs to be split up with λ_2 multiplied by a matrix with 1s where two vertices are connected by a path of length 2, and then μ times the rest of the matrix. For now, let's just define these, and then we'll try to make them clearer. Let A_d be the adjacency matrix with a 1 in spot (i, j) if $d(i, j) = d$ and a 0 otherwise. Note $A_0 = I$ and $A_1 = A$.

For example, the strongly regular condition above could be written

$$A^2 = kA_0 + \lambda A_1 + \mu(A_2 + A_3 + \dots)$$

But since

$$A_0 + A_1 + A_2 + \cdots = J,$$

we have

$$A_2 + A_3 + A_4 + \cdots = J - A_1 - A_0 = J - A - I$$

Then for a radial graph $G = radial(v, k, \lambda(d))$, we have

$$A^2 = kA_0 + \lambda(1)A_1 + \lambda(2)A_2 + \lambda(3)A_3 + \dots$$

In fact, fixing $\lambda(0) = k$, as the number of common neighbors of a vertex with itself is its degree. We can succintly state this as

$$A^2 = \sum_{d=0}^{diam(G)} \lambda(d)A_d$$

An eigenvector x here is a configuration on our graph, so its elements sum to zero, which gives $Jx = 0$. We also have $Ax = \rho x$ aand $Ix = x$. But we hit a little roadblock now, where it went fine with strongly regular graphs.

$$A^2x = \sum_{d=0}^{diam(G)} \lambda(d)A_d x$$

$$\rho^2 x = kx + \lambda_1 \rho x + \sum_{d=2}^{diam(G)} \lambda(d)A_d x$$

These later sums seem a bit harder to characterize. How does x being an eigenvector of A affect $A_d x$?

It seems tough. Let's approach it in a different way. For a graph, the following three are equivalent:

1. G is strongly regular with parameters (v, k, λ, μ) for some integers k, λ, μ.

2. $A^2 = (\lambda - \mu)A + (k - \mu)I +$ for certain real numbers k, λ, μ

3. G has exactly two distinct restricted eigenvalues.

Let's go the eigenvalue route. Coding up $H_2(g)$ is easy:

```
def H2(g):
    V = Subsets(list(range(1,2*g)),g)
    E = [(S,T) for S in V for T in V if len(S.symmetric_difference(T)) == 2]
    G = Graph(E)
    return G
```

Let's go ahead and use this to plot some bigger graphs:

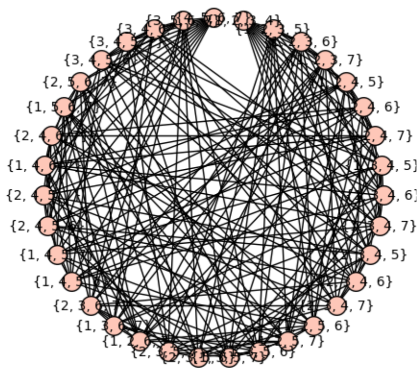


Figure 5: $H_2(4)$

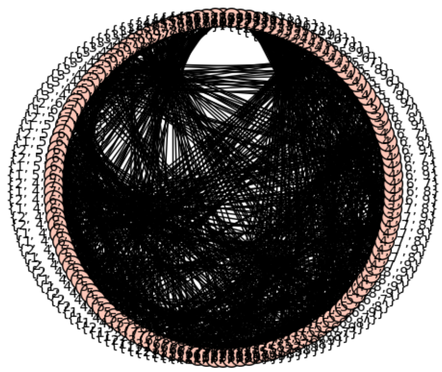


Figure 6: $H_2(5)$

Here are the eigenvalues for each $H_2(g)$:

$$H_2(2) \ : \ [2, (-1)^2]$$

$$H_2(3) \ : \ [6, 1^4, (-2)^5]$$

$$H_2(4) \ : \ [12, 5^6, 0^{14}, (-3)^{14}]$$

$$H_2(5) \ : \ [20, 11^8, 4^{27}, (-4)^{42}, (-1)^{48}]$$

$$H_2(6) \ : \ [30, 19^{10}, 10^{44}, 3^{110}, (-5)^{132}, (-2)^{165}]$$

So it doesn't seem like we have bounded number of eigenvalues. Let's find the elementary divisors and see if we can find a pattern in the critical group decomposition.

$$\mathcal{K}(H_2(2)) = \mathbb{Z}/3\mathbb{Z}$$

$$\mathcal{K}(H_2(3)) = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times (\mathbb{Z}/40\mathbb{Z})^3$$

$$\mathcal{K}(H_2(4)) = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z} \times (\mathbb{Z}/180\mathbb{Z})^8 \times (\mathbb{Z}/1260\mathbb{Z})^5$$

$$\mathcal{K}(H_2(5)) = (\mathbb{Z}/4\mathbb{Z})^{13} \times (\mathbb{Z}/12\mathbb{Z})^8 \times (\mathbb{Z}/36\mathbb{Z}) \times (\mathbb{Z}/252\mathbb{Z})^4 \times (\mathbb{Z}/504\mathbb{Z})^{17} \times (\mathbb{Z}/2016\mathbb{Z})^{26}$$

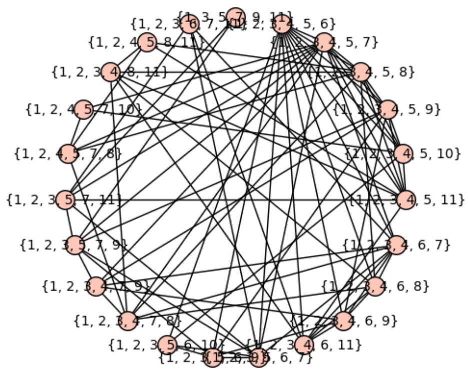
$$\mathcal{K}(H_2(6)) = [4^{44}, 16, 32^{31}, 160^2, 1120^{21}, 10080, 30240^{66}, 151200^{34}, 1663200^9]$$

Let's compare this with the critical groups of $\mathcal{S}_g$.

$$\mathcal{K}(\mathcal{S}_2) = \{e\}$$

$$\mathcal{K}(\mathcal{S}_3) = \mathbb{Z}/8\mathbb{Z}$$
$$\mathcal{K}(\mathcal{S}_4) = \mathbb{Z}/506\mathbb{Z}$$
$$\mathcal{K}(\mathcal{S}_5) = \mathbb{Z}/3642303\mathbb{Z}$$
$$\mathcal{K}(\mathcal{S}_6) = \mathbb{Z}/113910117019487 48\mathbb{Z}$$

I wonder if it's a coincidence that these are all cyclic? Here is $\mathcal{S}_6$:



I swear I've read a paper relating some property of the graph (cycles of some type?) to the number of invariant factors. I've searched for awhile and found a single paper, Cyclic Critical Groups of Graphs, which sounds spot on.

They talk about the conjecture I originally half-remembered: Wagner (Conj 4.2) conjectured in 2000 that a random graph will have cyclic critical group with probability 1. In 2014, Melanie Matchett Wood showed this was false! And that paper is a goldmine for understanding when a critical group might be cyclic.

And a beautiful number-theoretic result is that the probability a critical group is cyclic is at most

$$\frac{1}{\zeta(3)} * \frac{1}{\zeta(5)} * \frac{1}{\zeta(7)} * \cdots \approx 0.7935212$$

via studying Sylow p-subgroups of our critical group. Unfortunately, that is a very dense paper and I can't quite parse it right now.

It seems that these graphs tend to have pretty small automorphism groups. For example, $\mathcal{S}_6$ has a trivial automorphism group. Formally, a graph automorphism is a bijection $\phi : V \to V$ so that $(u, v) \in E$ if and only if $(\phi(u), \phi(v)) \in E$. This would mean we'd need a permutation ϕ of $\mathcal{N}_6$ so that whenever there exists $(x, y) \in S^c \times S$ so that $T = S \cup \{x\} - \{y\}$, we also have $(x', y') \in \phi(S)^c \times \phi(S)$ so that $\phi(T) = \phi(S) \cup \{x'\} - \{y'\}$.

But just from a graph-theoretic viewpoint, automorphisms of a graph preserve degree, so let's look at the degree sequence of $\mathcal{S}_6$.

$$[13, 12, 10, 9, 9, 9, 8, 7, 7, 7, 7, 7, 7, 6, 6, 6, 6, 5, 4, 4, 4, 3, 2]$$

So any automorphism must fix the vertices of degree $13, 12, 10, 8, 5, 3, 2$. It then must permute the semigroups

$$\textit{Degree } 4: \ \{1,2,3,6,7,11\}, \{1,2,4,5,8,11\}, \{1,2,3,5,6,10\}$$

$$\textit{Degree } 6: \ \{1,2,3,4,8,11\}, \{1,2,3,5,7,11\}, \{1,2,3,4,7,9\}, \{1,2,3,5,6,9\}$$

$$\textit{Degree } 7: \ \{1,2,3,5,7,9\}, \{1,2,3,4,7,8\}, \{1,2,3,4,6,11\},$$

$$\{1,2,3,4,6,9\}, \{1,2,3,4,6,8\}, \{1,2,3,4,5,10\}$$

$$\textit{Degree } 9: \ \{1,2,3,4,6,7\}, \{1,2,3,4,5,11\}, \{1,2,3,4,5,9\}$$

Let's consider these vertices now. The neighborhood of a vertex will be denoted $N_G(v)$. Here are the neighborhoods of the vertices of degree 4:

$$N_{\mathcal{S}_6}(\{1,2,3,6,7,11\}) = [\{1,2,3,5,7,11\}, \{1,2,3,4,6,7\}, \{1,2,3,4,6,11\}, \{1,2,3,5,6,7\}]$$

$$\textit{Degrees } = [4,9,7,8]$$

$$N_{\mathcal{S}_6}(\{1,2,4,5,8,11\}) = [\{1,2,4,5,7,8\}, \{1,2,3,4,5,8\}, \{1,2,3,4,8,11\}, \{1,2,3,4,5,11\}]$$

$$\textit{Degrees } = [5,10,6,9]$$

$$N_{\mathcal{S}_6}(\{1,2,3,5,6,10\}) = [\{1,2,3,4,5,6\}, \{1,2,3,4,5,10\}, \{1,2,3,5,6,7\}, \{1,2,3,5,6,9\}]$$

$$\textit{Degrees } = [12,7,8,6]$$

As the degree sequence of the neighborhoods are all different, none can be mapped to each other, so all vertices of degree 4 must also be fixed. For the vertices of degree 9, the neighborhood degree sequences are

$$[4,6,7,7,7,7,8,12,13]$$

$$[4,6,6,7,7,9,10,12,13]$$

$$[6,6,7,7,7,9,10,12,13]$$

so they must all be fixed. For degree 6, our neighborhood degree sequences are

$$[4,7,7,7,9,10]$$

$$[2,4,7,8,9,13]$$

$$[7,7,7,9,9,13]$$

$$[4,7,7,8,9,12]$$

For degree 7, our neighborhood degree sequences are

$$[2,6,6,6,8,9,13]$$

$$[5,6,6,7,9,10,13]$$

$$[4,6,7,7,9,9,12]$$

$$[6,6,7,7,9,9,12]$$

$$[6, 7, 7, 7, 9, 10, 12]$$

$$[3, 4, 9, 9, 10, 12, 13]$$

Nice! So no automorphisms.

Another natural question would be about Hamiltonian cycles. A graph contains a *Hamiltonian Path* if it has a path going through each vertex exactly once. It has a *Hamiltonian Cycle* if you can find a Hamiltonian Path that starts and ends at the same place. A graph with a Hamiltonian cycle is called a *Hamiltonian Graph*.

An important and concise theorem of Oystein Ore in 1960 states that a graph G contains a Hamiltonian cycle if for all pairs u, v of non-adjacent vertices, we have $deg(u) + deg(v) \geq n$. Dirac had a characterization in 1952, but the apparent best characterization we have today is the Bondy-Chvatal Theorem in 1976, which says that G has a Hamiltonian cycle if and only if its *Hamiltonian closure* does.

Let's stick with Ore's Theorem and see (1) whether $H_2(g)$ satisfies the condition and (2) what $\lambda(d)$ needs to look like for a radial graph to be Hamiltonian. As an example of this, the study of Strongly Regular Hamiltonian graphs is a very rich area of research, but Ore's Theorem gives a baseline. If G is a k-regular graph, then for any pair of vertices, $deg(u) + deg(v) = 2k$, so if G has $\leq 2k$ vertices, we know G is Hamiltonian.

Bill Jackson shows that we can improve this bound to $n \leq 3k$ implies Hamiltonian if G is 2-connected.

Dozens of more specific cases have been studied. Hamiltonian paths most classically pop up in the traveling salesman problem, which is NP-Complete. But let's look at $H_2(g)$. For $g = 2, 3$, Ore's Theorem does indeed guarantee us a Hamiltonian cycle.

Bill Jackson's result allows us to say $H2(g)$ is also Hamiltonian for $g = 4, 5$. For $g = 6$, it doesn't, but the edge-connectivity of $H2(g)$ seems to be equal to the degree $g(g-1)$, which I have to imagine forces a Hamiltonian cycle. In fact, in such a case, the greedy algorithm should guarantee us a Hamiltonian cycle.

It seems not! In this paper, Zhan gives a good introduction to the subject. The example of $K_{n,n+1}$ gives an n-connected non-Hamiltonian graph, with minimum degree $n-1$. He mentions the conjecture that every 4-connected line graph is Hamiltonian, and proves that every 7-connected line graph is Hamilton-Connected, meaning any two vertices are connected by a Hamiltonian path.

The Bondy-Chvatal theorem can actually be extended to determine whether G is Hamilton-Connected: If all non-adjacent vertices u and v have $deg(u) + deg(v) \geq n+1$, then G is Hamilton-Connected if and only if $G+uv$ is Hamilton-Connected.

And this paper by Goldsmith and White explicitly look for graphs with minimum degree equal to edge connectivity. Let's see if this lets us say anything about $H_2(g)$. There is a theorem of Chartrand that says that if the minimum degree of G is at least $n/2$, then it must contain a Hamiltonian cycle. Although this is good, it doesn't help us with $H_2(g)$, and neither do their other results.

Another paper that we have to look at is a note by Bollobas. Even the first paragraphs clarifies a lot. If G has edge-connectivity λ, then we can partition $V = U \sqcup W$ so that we have exactly λ edges between U and W. If we're trying to maximize such graphs, V and W must be complete graphs.

By analyzing the degrees of vertices in each part, Bollobas finds a condition on the degree sequence of G to guarantee a Hamiltonian cycle. He notes that this could be used to describe maximal degree sequences that don't have edge-connectivity equal to minimum degree. But he uses it to give sufficient conditions that they are equal. Unfortunately, this doesn't help us since we're dealing with a complete graph.

Perhaps a more personalized approach could help. A thesis by Harney gives a nice history of the topic of the Hamming graph $H_q(n,d)$, where our vertices come from $(\mathbb{Z}/q\mathbb{Z})^d$ and are connected if their Hamming distance is at least d. I'm going to screenshot a part of one page:

Proposition 2.2.4.
(a) $H_q(n,d)$ is simple.
(b) $H_q(n,d)$ is connected.
(c) $H_q(n,d)$ is regular of degree $\sum_{i=0}^{n-d} \binom{n}{i}(q-1)^{n-i}$.
(d) $H_q(n,d)$ is vertex-transitive.
(e) $H_q(n,d)$ is the undirected Cayley graph of the group $(\mathbb{Z}_q^n, +)$ and the subset $\mathcal{S}_d = \{v \in \mathbb{Z}_q^n \mid wt(v) \geq d\}$, which generates the group.

Proof. (e) Since $(q,d) \neq (2,n)$, the set $\mathcal{S}_d$ indeed generates the group $\mathbb{Z}_q^n$. In particular, the set of vectors $B = \{\mathbb{1}+e_1, \mathbb{1}+e_2, \ldots \mathbb{1}+e_{n-1}, \mathbb{1}\} = \{b_1, \ldots, b_n\}$ form a basis of the $\mathbb{Z}_q$-module $\mathbb{Z}_q^n$ and are in $\mathcal{S}_d$ for any admissible d. To see that B is indeed a basis, note that all the standard basis vectors e_i are in the span of B. Thus B generates $\mathbb{Z}_q^n$. A simple cardinality argument shows that they are linearly independent. By the definition of the Hamming graphs, $x, y \in \mathbb{Z}_q^n$ are adjacent in $H_q(n,d)$ iff $y = x + v$ for some $v \in \mathcal{S}_d$. This shows that the graph is the stated Cayley graph.
(a) This follows from (e) since $0 \notin \mathcal{S}_d$.
(b) Since $\mathcal{S}_d$ generates $\mathbb{Z}_q^n$, (b) follows immediately from (e).
(c) For any undirected Cayley graph, the regularity is simply the size of the generating set, in our case $|\mathcal{S}_d|$. Thus we simply need to count the number of vectors in $\mathbb{Z}_q^n$ with weight at least d. We sum over the number of coordinates with an entry of 0, and immediately get that $|\mathcal{S}_d| = \sum_{i=0}^{n-d} \binom{n}{i}(q-1)^{n-i}$.
(d) All Cayley graphs are vertex-transitive. □

Definition 2.2.5. A cubelike graph is a Cayley graph in which the underlying group is the elementary abelian group $\mathbb{Z}_2^n$ for some $n \in \mathbb{N}$.

Corollary 2.2.6. *$H_2(n,d)$ is a cubelike graph.*

First, sorry for the rough notation overlap...Second, the function $wt(v)$ is the Hamming distance of v from 0. And we'll see how the perspective of this as a Cayley graph will be helpful in a moment.

Onto Hamiltonicity of $H_q(n,d)$, Harney gives a few more nice characterizations of properties of the graph before giving the big result.

Corollary 1 (Harney) *If $d = 1$, then $diam(H_q(n,d)) = 1$. If $d > 1$ and*

$q \geq 3$, then $diam(H_q(n,d)) = 2$. If $q = 2$ and $d > 1$, then letting

$$A = \left\lceil \frac{\lceil \frac{d}{2} \rceil}{2(n-d)} \right\rceil,$$

we have

$$A \leq diam(H_q(n,d)) \leq A + 1$$

Conjecture 2 *All finite connected Cayley graphs on at least three vertices are Hamiltonian.*

Theorem 2 *Suppose G is a finite abelian group of order at least 3. Then $Cay(G,S)$ is Hamiltonian for all generating set S.*

Corollary 2 *All Hamming Graphs $H_q(n,d)$ are Hamiltonian.*

He proceeds to prove various strengthened versions of this corollary. But considering our graph $H_2(g)$ was defined to be (in Harney's notation) the Hamming graph $H_{2g-1}(g,2)$, we get that it must be Hamiltonian for all g. Great!

2.1 Numerical Semigroups by Genus For Easy Use in Sage

As the purpose of these documents is pretty much to keep information I might later forget, I'm going to go ahead and post the sets of numerical semigroups for the graph in a way that's easily moved back to sage to form the graph, since the COS returns numerical semigroups in an unhelpful way.

```
Genus 1: Set([1])
Genus 2: Set([1,2]), Set([1,3])
Genus 3: Set([1,2,3]), Set([1,2,4]), Set([1,2,5]), Set([1,3,5])
Genus 4: Set([1,2,3,4]), Set([1,2,3,5]), Set([1,2,3,6]),
Set([1,2,3,7]), Set([1,2,4,5]), Set([1,2,4,7]), Set([1,3,5,7])

Genus 5: Set([1,2,3,4,5]), Set([1,2,3,4,6]), Set([1,2,3,4,7]),
Set([1,2,3,4,8]), Set([1,2,3,4,9]), Set([1,2,3,5,6]),
Set([1,2,3,5,7]), Set([1,2,3,5,9]), Set([1,2,3,6,7]),
Set([1,2,4,5,7]), Set([1,2,4,5,8]), Set([1,3,5,7,9])

Genus 6: Set([1,2,3,4,5,6]), Set([1,2,3,4,5,7]),
Set([1,2,3,4,5,8]), Set([1,2,3,4,5,9]), Set([1,2,3,4,5,10]),
Set([1,2,3,4,5,11]), Set([1,2,3,4,6,7]), Set([1,2,3,4,6,8]),
Set([1,2,3,4,6,9]), Set([1,2,3,4,6,11]), Set([1,2,3,4,7,8]),
Set([1,2,3,4,7,9]), Set([1,2,3,4,8,9]), Set([1,2,3,5,6,7]),
Set([1,2,3,5,6,9]), Set([1,2,3,5,6,10]), Set([1,2,3,5,7,9]),
Set([1,2,3,5,7,11]), Set([1,2,3,6,7,11]), Set([1,2,4,5,7,8]),
Set([1,2,4,5,7,10]), Set([1,2,4,5,8,11]), Set([1,3,5,7,9,11])
```

```
Genus 7:
Set([1,2,3,4,5,6,7]), Set([1,2,3,4,5,6,8]), Set([1,2,3,4,5,6,9]),
Set([1,2,3,4,5,6,10]), Set([1,2,3,4,5,6,11]), Set([1,2,3,4,5,6,12]),
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Set([1,2,3,4,5,7,10]), Set([1,2,3,4,5,7,11]), Set([1,2,3,4,5,7,13]),
Set([1,2,3,4,5,8,9]), Set([1,2,3,4,5,8,10]), Set([1,2,3,4,5,8,11]),
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Set([1,2,3,4,6,7,8]), Set([1,2,3,4,6,7,9]), Set([1,2,3,4,6,7,11]),
Set([1,2,3,4,6,7,12]), Set([1,2,3,4,6,8,9]), Set([1,2,3,4,6,8,11]),
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Set([1,2,3,5,7,9,11]), Set([1,2,3,5,7,9,13]), Set([1,2,4,5,7,8,10]),
Set([1,2,4,5,7,8,11]),Set([1,2,4,5,7,10,13]), Set([1,3,5,7,9,11,13])

Genus 8: Set([1,2,3,4,5,6,7,8]),Set([1,2,3,4,5,6,7,9]),
Set([1,2,3,4,5,6,7,10]),Set([1,2,3,4,5,6,7,11]),Set([1,2,3,4,5,6,7,12]),
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Set([1,2,3,5,7,9,11,15]),Set([1,2,4,5,7,8,10,11]),Set([1,2,4,5,7,8,10,13]),
Set([1,2,4,5,7,8,11,14]),Set([1,3,5,7,9,11,13,15])
```

2.2 Back to $H_2(g)$ and $\mathcal{S}_g$

The chromatic numbers for $H_2(g)$ for $g = 2, 3, 4, 5, 6$ are

$$\chi(H_2(g)) = 3, 5, 6,$$

and the chromatic numbers for $\mathcal{S}_g$ are

$$\chi(\mathcal{S}_g) = 2, 3, 4, 5, 6, 7$$

Seeing the apparent pattern of $\chi(\mathcal{S}_g) = g$, let's go ahead and compute their chromatic polynomials. We'll abbreviate this as $\chi_g(x)$.

$$\chi_2(x) = x(x-1)$$
$$= x^2 - x$$
$$- - - - - - - - -$$
$$\chi_3(x) = x(x-1)(x-2)^2$$
$$= x^4 - 5x^3 + 8x^2 - 4x$$
$$- - - - - - - - -$$
$$\chi_4(x) = x(x-1)(x-2)^2(x-3)(x^2-5x+7)$$
$$= x^7 - 13x^6 + 70x^5 - 199x^4 + 313x^3 - 256x^2 + 84x$$
$$- - - - - - - - -$$
$$\chi_5(x) = x(x-1)(x-2)^2(x-3)(x-4)(x^2-6x+12)(x^4-13x^3+65x^2-149x+133)$$
$$x^{12} - 31x^{11} + 438x^{10} - 3720x^9 + 21076x^8 - 83499x^7 + 235545x^6 - 471842x^5 +$$
$$+ 655528x^4 - 598920x^3 + 322032x^2 - 76608x$$

The chromatic polynomial has a varieties of properties, which we'll state in terms of $\mathcal{S}_g$:

1. The degree of $\chi_g(x)$ is $N(g)$.
2. All coefficients of $\chi_g(x)$ are non-zero except the constant term, which is 0.
3. The absolute value of the coefficient of $x^{N(g)-1}$ is the number of edges of $\mathcal{S}_g$.

June Huh worked with hypersurfaces for his thesis at University of Michigan and proved the wonderful result that the coefficients of the chromatic polynomial are log-concave, which was a long-standing conjecture called Read's Conjecture. That link is his paper on the coefficients of the chromatic polynomial and it's honestly an incredibly well-structured and clear paper! Highly suggested.

Diving into that result brought me to the world of Matroids and Cyclic Sieving, and sizes of Cohomology groups, Hodge Theory, which I found incredible, but I won't (and couldn't even if I wanted to!) ramble about that now. I also just found that June Huh did a video with Numberphile on this.

Here are the number of edges of $\mathcal{S}_g$ for $g = 2, 3, 4, 5, 6, 7$.

$$1, 5, 13, 31, 79, 173$$

which does not appear in the OEIS.

What are the coefficients of the chromatic polynomial, then? Well often, we start with deletion-contraction, which is what led Tutte to defining the Tutte polynomial we discussed before. A function satisfies the deletion-contraction condition if

$$f(G) = f(G - e) + f(G/e)$$

for all edges. Deletion-contraction properties are great, but I want to look for a combinatorial interpretation of the coefficients.

First, by definition

$$\chi_G(x) = \sum_{S \subset E} (-1)^{|S|} x^{c(S)},$$

where $c(S)$ is the number of connected components of the subgraph induced by S. Another fun fact is

$$\chi'_G(1) \geq 0$$

for any connected graph G, with strict inequality if biconnected.

And one of my favorite results in math is that $\chi_G(-1)$ counts the number of acyclic orientations of G. And this is pretty much the inspiration for the cyclic sieving phenomenon. As usual in this area, the proof relies on applying deletion-contraction repeatedly and then concluding the fact holds for the trivial graph.

I found it! I learned about the connection between the chromatic polynomial and *hyperplane arrangements*, which is absolutely beautiful, with this paper by. Bruce Sagan, who introduced me to combinatorics in my first (and only) REU at Michigan State University.

Another wonderful thing Sagan did with Andreas Blass was to give a bijection between coefficients of the chromatic polynomial and "broken circuits". I also want to mention here an amazing bijection Blass used to prove A bijection between finite binary trees and 7-tuples of such trees, using a beautiful combinatorial interpretation of algebraic equations called Combinatorial Species.

The wiki page isn't terribly helpful, but here is one paper that I think is. This and the *cycle index series* are ways to understand labelled and unlabelled combiantorial objects using generating functions. For example, this allows one to give a generating function for the number of graphs on n vertices *up to isomorphism*!

3 Actually back to $\mathcal{S}_g$ (or something)

If we take a numerical semigroup S and choose a pair (x, y) from $S^c \times (S \cap [2g-1])$ at random, then what is the probability that $S \cup \{x\} - \{y\}$ is a numerical semigroup? The number of pairs is $g(2g-1-g) = g(g-1)$. Then $deg(S)$ in $\mathcal{S}_g$ is the number of successes, so the probability is

$$\frac{deg(S)}{g(g-1)}$$

Averaging over $S \in \mathcal{N}_g$ then gives the average probability that a swap of S is a numerical semigroup.

$$\frac{1}{N(g)} \sum_{S \in \mathcal{N}_g} \frac{deg(S)}{g(g-1)} = \frac{1}{N(g)g(g-1)} \sum_{S \in V(\mathcal{S}_g)} deg(S) = \frac{2|E|}{N(g)g(g-1)}$$

Here are those values for $g = 2, 3, 4, 5, 6, 7$:

$$\frac{1}{2}, \frac{5}{12}, \frac{13}{42}, \frac{31}{120}, \frac{79}{345}$$

Definitely seems to be going to 0, which makes sense in my mind.

Let's compute the average degrees of $\mathcal{S}_g$ for $g = 2, 3, \ldots$.

$$1, \frac{5}{2}, \frac{26}{7}, \frac{31}{6}, \frac{158}{23}, \frac{346}{39}$$

In decimals,

$$1, \; 2.5, \; 3.7, \; 5.2, \; 6.9, \; 8.9$$

I want to go back to that idea of a *radial graph* and just look at if any general properties are deducible or if it's too general a type of graph. Which leads to question one.

Question 1 *Asymptotically, what is the probability a graph is a radial graph?*

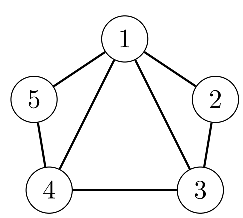
Remember we're calling a graph G *radial* if there exists a function

$$\lambda : \{0, 1, \ldots, diam(G)\} \to \mathbb{N}$$

so that any pair of vertices u and v have $\phi(d(u, v))$ common neighbors.

Question 2 *Given a graph, how can we decide whether it is radial and what's the complexity?*

Non-radial graphs are easy to construct.



Then $d(1, 2) = 1$ and $d(1, 3) = 1$ but 1 and 3 have two common neighbors and 1 and 2 have one.

I'll make a note here that as we mentioned before, another natural generalization of strongly regular graphs would be looking at graphs with a fixed number of eigenvalues. Lots of people have done this: Ex 1, Ex 2, Ex 3.

So a natural object to consider would be the *distance matrix* $D(G)$ of G, where the $(i,j)^{th}$ spot is $d(i,j)$. This collects the information of all A_d (where we have 1 if $d(u,v) = d$ and 0 otherwise). Here, we can construct A_d from D by continually subtracting 1, and if appears on spot (i,j) after subtracting k times, add a 1 to spot (i,j) and (j,i) in A_k.

Starting with $D(G)$, we have zeros only along the diagonal, with shows $A_0 = I$. Then $D - J$ will have zeros only when the corresponding entry in $A(G)$ is 1. The diagonals will be -1, and the other spots will be ≥ 1. Similarly,

$$\textit{For any } k,\ (D - kJ)[i,j] = 0 \iff A_k[i,j] = 1$$

Not all circulant graphs are strongly regular - C_6 for example. And it fails because we could choose vertices 1 and 3 that are non-adjacent with one neighbor and at the same time choose 1 and 4 that are non-adjacent but share no neighbors.

But $d(1,3) = 2$ while $d(1,4) = 3$, so the issue is resolved by broadening to radial graphs. The adjacency matrices of C_6 are:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

So we should have

$$A^2 = 2I + 0A + 1A_2 + 0A_3 = 2I + A_2 = \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix}$$

And we see C_6 is a radial graph with $\phi(d) = 2, 0, 1, 0, \ldots$. This is pretty easy to implement into Sage:

```
n=6
G = graphs.CycleGraph(n)
D = G.distance_matrix()
J = matrix.ones(n)
AdList = []
for k in range(G.diameter()+1):
    Ak = matrix.zero(n)
    for i in range(n):
        for j in range(i,n):
            if (D - k*J)[i,j] == 0:
                Ak[i,j] = 1
```

```
            Ak[j,i] = 1
    AdList.append(Ak)
for A in AdList:
    print('A_{}={}'.format(AdList.index(A),show(A)))
    print('--')
```

Question 3 *Given a sequence $a_1, a_2, \ldots$ of non-negative integers that is eventually* 0*, does there exist a radial graph with $\phi(i) = a_i$? If it doesn't stabilize to* 0*, can we construct an infinite radial graph?*

I think it's time to introduce NAUTY. I will be using it through the Sage interface, and will put down how here (it's very simple). Here is an example I grabbed from some work 5 years ago! I have no idea what exactly I was looking for.

```
K = graphs.CompleteBipartiteGraph(5,5)
Kp = K.plot(layout='circular')
Kp.show()
Ke = K.edges()
print
gen = graphs.nauty_geng("10 -c")
for G in gen:
    if G.size() == 26:
        p = G.plot(layout='circular')
        p.show()
        G.triangles_count()
        G.delete_edges(Ke)
        pnew = G.plot(layout='circular')
        pnew.show()
```

The important bit there is ***gen = graph.nauty_geng("10 -c")***. That command generates all graphs on 10 vertices that are connected. Removing the $-c$ removes the connected requirement, and we can add other parameters to get more specific families of graphs. This can be found in SageMath Documentation.

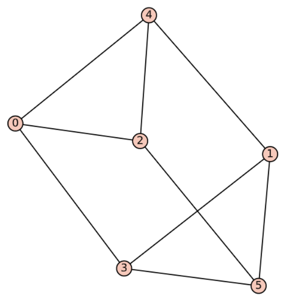
Or with no parameters, we can simply generate all graphs with 3 vertices. If we want to restrict to regular graphs, we could throw in a single line:

```
gen = graphs.nauty_geng("3")
for G in gen:
    if len(Set(G.degree_sequence())) == 1:
        G.show()
```

Wonderful! Searching over connected, regular graphs, the first time

$$A^2 = \sum_{d=0}^{diam(G)} \lambda(d) A_d$$

fails is for the following graph:




```
A_0=None
lk = 3
--
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

```
A_1=None
lk = 1
--
```

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

```
A_2=None
lk = 2
```

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

with

$$A^2 = \begin{bmatrix} 3 & 2 & 1 & 0 & 1 & 2 \\ 2 & 3 & 2 & 1 & 0 & 1 \\ 1 & 2 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 2 & 1 \\ 1 & 0 & 1 & 2 & 3 & 2 \\ 2 & 1 & 0 & 1 & 2 & 3 \end{bmatrix} \quad and \quad \sum_{d=0}^{diam(G)} \lambda(d) A_d = \begin{bmatrix} 3 & 2 & 1 & 1 & 1 & 2 \\ 2 & 3 & 2 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 & 1 & 1 \\ 1 & 1 & 2 & 3 & 2 & 1 \\ 1 & 1 & 1 & 2 & 3 & 2 \\ 2 & 1 & 1 & 1 & 2 & 3 \end{bmatrix}$$

You can see the only difference is one diagonal overcounting by 1. Whether it overcounts or undercounts, if it's not equal, we know it's not radial. The issue being it's not distance-1 regular. Vertices $(1, 4)$ have no common neighbors but $(1, 5)$ have one.

So radial graphs lie in the middle of regular graphs and strongly regular graphs. Here are the number of regular graphs that that are not radial.

$$n = 6: \qquad 1\ graph \qquad \frac{1}{5} \approx 0.2$$

$$n = 7: \qquad 2\ graphs \qquad \frac{2}{4} \approx 0.5$$

$$n = 8: \qquad 12\ graphs \qquad \frac{12}{17} \approx 0.706$$

$$n = 9: \qquad 18\ graphs \qquad \frac{18}{22} \approx 0.818$$

$$n = 10 : \qquad 160\ graphs \qquad \frac{160}{167} \approx 0.958$$

Note the number of connected regular graphs on n vertices is A005177 on OEIS. Probabilistic arguments give an expected diameter of an r-regular graph as

$$\left\lceil \frac{2rnln(n)}{ln(r-1)} + 1 \right\rceil$$

It seems a very large amount of the work done on random regular graphs can be attributed Bela Bollobas. One particular interesting bit is Friedman's Theorem that most regular graphs are Ramanujan graphs!

A survey of Ramanujan graphs by Murty covers the topic well. It also reminds me that's the book I first saw the Paley graphs out of! Murty and Bondy's Graph Theory, which I mentioned in the last document - and I'll add an update there - page 116.

Given the above data, I'd make a conjectural answer to Question 1:

Conjecture 3 *The probability a random regular graph is radial is* 0*.*

Let's count how many radial graphs are strongly regular, to see a relative density:

$$n = 6 : \qquad 2\ graphs \qquad \frac{2}{4} \approx 0.5$$

$$n = 7 : \qquad 0\ graphs \qquad \frac{0}{2} \approx 0.0$$

$$n = 8 : \qquad 2\ graphs \qquad \frac{2}{5} \approx 0.4$$

$$n = 9 : \qquad 2\ graphs \qquad \frac{2}{4} \approx 0.5$$

$$n = 10 : \qquad 4\ graphs \qquad \frac{4}{7} \approx 0.57$$

It feels like I've never had a better excuse to actually read about Ramanujan graphs and get a better feel for their connection to number theory. Using the Murty survey, I'm amazed very early on! Here's a really cool connection:

Theorem 3 *Let G be a finite abelian group and S a symmetric subset of size k. Then the eigenvalues of the adjacency matrix of $X(G, S)$ are given by*

$$\lambda_\chi = \sum_{s \in G} \chi(s)$$

for each irreducible character χ of G.

Where the definition of the *Cayley Graph* $X(G, S)$ is similar to the other definitions we've encountered: The vertex set is the elements of the group G and we draw edges by right-multiplying by the elements $s \in S$. And the symmetric assumption just means $s \in S \Leftrightarrow s^{-1} \in S$, so the graph is undirected, but this actually applies to digraphs as well.

Ramanujan graphs are defined as those regular graphs whose largest eigenvalue is $\lambda \leq 2\sqrt{d-1}$. They can be defined in many more ways and appear in a lot of the things we've already discussed, like Moore graphs. The survey goes into results like Dirichlet's Determinant formula, counting non-backtracking walks in regular graphs, and the relation to a Riemann Hypothesis for the Ihara Zeta Function of a graph.

It also inspired the idea that we could perhaps find a recurrence for A_d. Remember that this is the matrix with a 1 in spot (u, v) if $d(u, v) = d$, and zero otherwise. Note that a particular spot (u, v) will be 1 in exactly one A_d, which shows

$$J = \sum_{d=0}^{diam(G)} A_d$$

Assigning a certain weight $\lambda(d)$ to each path of length d gives the weighted sum

$$J_\lambda = \sum_{d=0}^{diam(G)} \lambda(d) A_d$$

Interpreting the weight as ***The number of common neighbors of vertices distance d apart*** gives the radial graph definition, and gives

$$J_\lambda = A_1^2$$

I wonder if any other fun things pop up with different weights? We'd want to look at some property that is non-constant on paths of length d and see what graphs can exist if we fix it by choosing a $\lambda(d)$.

Or really, it's kind of the opposite, isn't it? By choosing $J_\lambda = A^2$, we gain the interpretation of the weights being the number of common neighbors of vertices distance d apart. Any interpretation will be of the form "BLAH for vertices distance d apart".

For example, choosing $J_\lambda = J$ gives the constant function $\lambda(d) = 1$. Hence the interpretation simply being an indicator function for being distance d apart. So

$$J = J_1$$

If we choose $J_\lambda = D$, the distance matrix, then we'd get $\lambda(d)$ being distance of the two vertices. In other words, the identity function $\lambda(d) = d$.

The classes of graphs satisfying these sums will be all of them, though. Choosing $J_\lambda = A^2$, as we've seen, reduces the class of graphs satisfying the sums to radial graphs.

Choosing $J_\lambda = Deg(G)$, the matrix with $deg(v)$ along the diagonal and 0 elsewhere, gives the sum

$$Deg(G) = \sum_{d=0}^{diam(G)} \lambda(d) A_d$$

which means we must have $\lambda(d) = 0$ for $d \geq 1$ and for $d = 0$, it can be any constant. Hence, the class of graphs satisfying this choice is the set of G such that

$$Deg(G) = kI \quad for\ some\ k$$

In other words, regular graphs!

Choosing $J_\lambda = A$ forces the weights $\lambda(d) = 0, 1, 0, 0, \ldots$, which naturally is just interpreted as "we only care about vertices that are connected", but we can also phrase this as *the number of common neighbors a vertex has with itself*, i.e. its neighbors. Choosing A^2 gives radial graphs, which gives the combinatorial interpretation that all vertices at distance d have the same number of common neighbors.

Before doing the next ones, I want to write down what might be a clearer way to think about this: ***Choosing a J_λ that encodes some property tells us to look at how vertices at different distances can contribute to that property. By asking which graphs satisfy the sums, we're asking about graphs whose contribution to that property depends only on the distance between vertices.***

What kind of graph arises if we want $J_\lambda = A^3$? The (u, v) entry is the number of paths of length 3 from u to v. If such a path goes from u back to itself, then it's a triangle.

Turning this info into matrices, it tells us the diagonals are the number of triangles containing that vertex, call it $t(v)$. Hence $\lambda(0)$ will tell us that every vertex must be contained in the same amount of triangles. If $\lambda(0) = 0$, then this amount is *no triangles*, meaning G is a triangle-free graph.

Which is awesome because one of the first graph theory things I wrote about in this document was on Turan's theorem about the maximum number of edges in a triangle-free graph. It'd be cool to connect edge count to the weights $\lambda(d)$.

If u and v are not connected, then the path connecting them can't backtrack since it's length 3. Then the path is $u - a_1 - a_2 - v$, so the number in spot (i, j) is the number of edges between $N(u)$ and $N(v)$. Which means fixing a $\lambda(d)$ for $d \geq 1$ requires this number to be determined only by the distance between the vertices. This seems like a very large restriction.

If you're thinking of a graph as representing a relation, then triangles in graphs represent transitivity of that relation for those three vertices. Although it's not always the first place my mind goes, there are many real-world applications of counting triangles in graphs. In communication networks or internet searching (i.e. Google's search algorithm), a triangle at a vertex could represent new material that is closely related to the current material, which is the hope of a good search algorithm. They go over many more applications immediately in that link.

They define the local clustering coefficient of a vertex as

$$C(v) = \frac{|\{(i,j) \in E \mid i, j \in N(v)\}}{\binom{deg(v)}{2}}$$

and the global clustering constant as the average of these

$$C(G) = \frac{1}{n} \sum_{v \in V} C(v)$$

Using the notation we defined above, the clustering coefficient is $C(v) = t(v)/\binom{deg(v)}{2}$.

Let's make a definition attaching graphs to choices of J_λ.

Definition 3 We'll say that a graph G is *represented by* a matrix M if there exists some choice of $\lambda(d)$ so that

$$M = \sum_{d=0}^{diam(G)} \lambda(d) A_d(G)$$

We'll write

$$Class(J_\lambda) = \{G\ represented\ by\ J_\lambda\}$$

and if we've chosen a specific $\lambda(d)$, we'll write this as

$$Class(J_\lambda, \lambda(d))$$

Rephrasing what we previously talked about, we have

1. The inspiration for all this:

$$Class(A^2) = \{radial\ graphs\}$$

$$Class(A^2, [k, a, b, b, \dots]) = \{strongly\ regular\ graphs\ (n, k, a, b)\}$$

2. Every graph is represented by J uniquely $\lambda(d) = 1$:

$$Class(J) = \{all\ graphs\}$$

$$Class(J, [1, 1, \dots]) = \{all\ graphs\}$$
$$Class(J, else) = \{\}$$

3. Every regular graph is represented by their degree matrix. Meaning

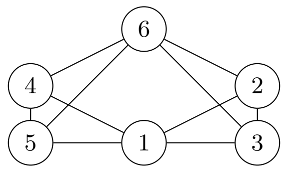
$$Class(kI) = \{k - regular\ graphs\}$$

$$Class(kI, [k, 0, \dots]) = \{k - regular\ graphs\}$$
$$Class(kI, else) = \{\}$$

   But the two are the same since if $J_\lambda = kI$, we must have $\lambda(d) = 0$ for all $d \geq 1$, so the λ is forced.

4. Every graph is represented by any of its adjacency matrices A_k, while $\lambda(k) = 1$ and $\lambda(d) = 0$ otherwise.

Then going back to the clustering coefficients, if $G \in Class(A^3)$, then we have must have the same amount of triangles going through every vertex. Meaning $t(v)$ is constant for all v, call it T. I think a graph with this property could be non-regular...Indeed!



We have

$$t(1) = |\{145, 123\}|$$
$$t(2) = |\{236, 123\}|$$
$$t(3) = |\{236, 123\}|$$
$$t(4) = |\{145, 456\}|$$
$$t(5) = |\{145, 456\}|$$
$$t(6) = |\{236, 456\}|$$

But $deg(6) = deg(4) = 4$ and all other vertices have degree 3.

But here's the question: Is that above graph actually in $Class(A^3)$? Remember there were more conditions than just the triangle one. But clearly we'd need $\lambda(0) = 4$. This graph's other adjacency matrices are:

$$A_1 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

And here is A^3:

$$A^3 = \begin{bmatrix} 4 & 9 & 9 & 9 & 9 & 4 \\ 9 & 4 & 5 & 4 & 4 & 9 \\ 9 & 5 & 4 & 4 & 4 & 9 \\ 9 & 4 & 4 & 4 & 5 & 9 \\ 9 & 4 & 4 & 5 & 4 & 9 \\ 4 & 9 & 9 & 9 & 9 & 4 \end{bmatrix}$$

So the answer is NO, and we can tell because, for example, spots $(1,3)$ and $(2,3)$ come from A_1 yet $A^3[1,3] = 9$ and $A^3[2,3] = 5$, so we can't have a single constant multiple of A_1 to contribute to A^3.

Question 4 *Is $Class(A^k)$ non-empty for all k?*

3.1 Reverse it

Let's see if we can get any benefit from a more algebraic point of view. If we interpret $\lambda(d)$ as coefficients c_i, then our sum is simply a linear combination of A_d. So if we consider each A_d a formal object, we can look at the formal ring

$$\mathbb{Z}[A_0, A_1, \ldots, A_{diam(G)}]$$

and each element of this ring "collapses" down to some matrix J_λ when we recognize the generators as matrices again. An "evaluation map" of sorts, which is helpful in Field theory for constructing isomorphisms of quotients to field extensions.

This inspiration comes from the Group Ring/Module and the beautiful study of Stanley-Reisner rings in algebraic combinatorics.

Though this can be defined more generally, for a graph G, the definition comes down to taking the formal ring $k[x_1, \ldots, x_n]$ over a field k and quotienting by the ideal I generated by the square-free monomials $\{x_i x_j \mid (i,j) \notin E\}$

Then the quotient $k[G] := k[x_1, \ldots, x_n]/I$ is called the Stanley-Reisner ring of the graph, and encodes the connections in the graph algebraically. From a commutative algebra perspective, we have a graded ring, for which we can compute a Hilbert series, the coefficients of which form the h-vector of G.

An absolutely wonderful presentation by Jose Alejandro Samper goes through the relevant Matroid language, shellable complexes, and then introduces Richard Stanley's h-vector Conjecture. They then relate it directly to chip-firing on the simplex and discuss special types of configurations of chips on the graph (stable, critical, supercritical,etc), which is something I didn't address when I previously discussed the chip-firing game and the critical group.

Conjecture 4 (Stanley) *The h-vector of a matroid is a pure O-sequence.*

That obviously would take some explaining, but I'm not going to do anything more with it and Samper already explained it wonderfully in those slides!

Going back to $\mathbb{Z}[A_0, \ldots, A_{diam(G)}]$, the "evaluation map" is

$$\mathcal{E}_G : \mathbb{Z}[A_0, \ldots, A_{diam(G)}] \to M_n(\mathbb{Z}),$$

given by actually multiplying the matrices by the coefficients and adding them. Then the question of whether a graph G is represented by a matrix M is asking whether $M \in Im(\mathcal{E}_G)$.

We have three immediate facts for any G:

- $J \in Im(\mathcal{E}_G)$ for all G.
- $A_d \in Im(\mathcal{E}_G)$ for all d.
- In particular, the zero matrix is in $Im(\mathcal{E}_G)$ for all G.

This is just restating the obvious facts about classes above.

As J is the identity matrix for the Hadamard Product, I think it'd be worthwhile to look at this as the product for our ring. This product is the element-wise multiplication of matrices and satisfies a ton of very interesting properties. We denote this by $A \odot B$.

In 1911, Schur showed that the Hadamard product of two positive-definite matrices is positive definite. But no adjacency matrices except the trivial one are positive definite, since $tr(A) = 0$ implies there must be at least one negative eigenvalue. Another fun fact is on the determinant:

$$det(A \odot B) \geq det(A)det(B)$$

And I just realized the Hadamard product of two A_d is always zero.

So let the ring $\mathbb{Z}[G]$ denote the ring of formal linear combinations of A_d with component-wise addition and Hadamard product. The Hadamard product is commutative, associative, and distributes over addition, which means it is a ring. And

$$(a_0 A_0 + \cdots + a_d A_d) \odot (b_0 A_0 + \cdots + b_d A_d)$$

will not have any cross-terms $A_i A_j$. And since $A_i \odot A_i = A_i$, the above product becomes

$$a_0 b_0 A_0 + \cdots + a_d b_d A_d$$

It's clear that

$$\mathcal{E}_G(X + Y) = \mathcal{E}_G(X) + \mathcal{E}_G(Y)$$

For the product, $\mathcal{E}_G(X \odot Y)$ has in spot (i,j) the product $a_k b_k$ where $k = d(i,j)$. In $\mathcal{E}_G(X) \odot \mathcal{E}_G(Y)$, the spot (i,j) has the product of $(i,j)^{th}$ spots from $\mathcal{E}_G(X)$ and $\mathcal{E}_G(Y)$. But the entry there is exactly a_k and b_k, where $k = d(i,j)$.

All together, this gives:

Proposition 1 *Let $M_n(\mathbb{Z})$ be ring of $n \times n$ integer matrices with the Hadamard product, and $Z[G]$ be defined as before for some graph G. The evaluation map*

$$\mathcal{E}_G : \mathbb{Z}[A_0, \ldots, A_{diam(G)}] \to M_n(\mathbb{Z})$$

is a ring homomorphism.

As each A_d is symmetric, we know $Im(\mathcal{E}_G) \subseteq Sym_n(\mathbb{Z})$. While not true for the usual product, it's clear that $Sym_n(\mathbb{Z})$ is closed under Hadamard product, so $Sym_n(\mathbb{Z})$ is a subring of $M_n(\mathbb{Z})$ under this product.

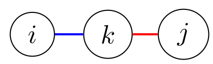
As a brief aside, let's interpret the Hadamard product of two adjacency matrices for graphs G, H. Then

$$(A(G) \odot A(H))[i,j] = 1 \Leftrightarrow A(G)[i,j] = A(H)[i,j] = 1 \Leftrightarrow (i,j) \in E(G) \cap E(H)$$

So the Hadamard product of adjacency matrices corresponds to taking the intersection of the two graphs. As a super aside, but a very cool fact involving that semiring $\mathbb{B}$ that we defined while on another aside about numerical semigroups and generalizations.

Remember that $\mathbb{B} = \{0, 1\}$ where $1+1 = 1$ was the new *prime semifield* that we got when going from rings to semirings. One cool thing about this is that in $M_n(\mathbb{B})$, addition of adjacency matrices corresponds to the union of the two graphs edge sets. Similarly, the symmetric difference of two graphs is encoded in the addition of adjacency matrices in $M_n(\mathbb{F}_2)$.

Regular multiplication of adjacency matrices will not be a $0, 1$- matrix, but we can interpret it similar to A^k for a single matrix. If G and H are two graphs, then $(A(G)A(H))[i, j]$ will be a sum over the i^{th} row of $A(G)$ and the j^{th} column of $A(H)$. It will collect a 1 whenever $(i, k) \in E(G)$ and $(j, k) \in E(H)$. This means we have a path the starts $i- > k$ on an edge of G and then ends on and edge $k- > j$. The blue edge is in G and the red edge is in H.



In general, the $(i, j)^{th}$ entry of $\prod_i A(G_i)$ is the number of paths from i to j whose t^{th} edge is in G_t.

Now that I wrote that, I am reminded of Γ_g, where we had red and blue edges but wanted to guarantee that every path we're interested in swapped between those two. Consider them as separate graphs (i.e. the "numerical semigroup part" and the "removing multiplicites" part) and the product of their adjacency matrices will tell us the number of alternating paths!

Let $F(g)$ be the part of Γ_g with blue edges and $M(g)$ be the part with red edges. The ordinarization number of S (defined by Bras-Amoros) is half the distance of the shortest alternating path from S to $\mathcal{O}_g$, hence it will be the smallest k for which the entry $(S, \mathcal{O}_g)$ in $(A(F(g))A(M(g)))^k$ is non-zero.

Since we know $\mathcal{E}_G$ is a ring homomorphism, we can study its kernel, which is an ideal in $\mathbb{Z}[A_0, \ldots, A_{diam(G)}]$. So what does an ideal look like with Hadamard product? An ideal is a subgroup $I \subset R$ that's a sponge: $rI \subseteq I$ for all $r \in R$. If we use the Hadamard product, this would mean $r \odot x \in I$ for all $x \in I$.

The $(i, j)^{th}$ entry of $r \odot x$ is $r_{ij}x_{ij}$. Since we can choose r to be the matrix with zeros everywhere except one spot, we know that if $x \in I$, then for each (i, j), the matrix with x_{ij} in spot (i, j) and zeros elsewhere must be in I.

Then I will consist of matrices so that each entry independently acts like a regular ideal in $\mathbb{Z}$. This allows us to say the following:

Proposition 2 *An ideal I in $M_n(\mathbb{Z})$ under Hadamard multiplication is non-trivial if and only if there is some (i, j) so that all $x \in I$ have $x[i, j] \neq \pm 1$.*

Proof 1 *If all (i, j) have some $x \in I$ with $x[i, j] = \pm 1$, then we have the matrix with only a 1 in that spot and 0 elsewhere. If we have that for all (i, j), then we have $J \in I$, so $I = R$.*

Conversely, if we have a (i, j) so that all $x \in I$ have $x[i, j] \neq \pm 1$, then we can never achieve a 1 in spot (i, j), and therefore $J \notin I$, so I is non-trivial.

Great, so now we can look back at $ker(\mathcal{E}_G)$. If $a_0A_0 + \cdots + a_dA_d \in ker(\mathcal{E}_G)$, then

$$a_0A_0 + \cdots + a_dA_d = J$$

But each matrix is multiplicatively independent, so this forces $a_i = 1$ for all i. And this shows $\mathcal{E}_G$ is injective.

That's fun, because it shows the following fact: if G is represented by J_λ, then the sequence λ is uniquely determined. But really the interest is on the image. It's isomorphic to $\mathbb{Z}[G]$ of course, but we want a better grasp on the actual elements in the image.

And of course, now it seems obvious that the only explanation is really that G is represented by M if and only if $M[i, j]$ is constant over all non-zero (i, j) in A_d, for each $d \geq 0$. Another way we could express this is that G is represented by M if and only if there exists $\lambda(d)$ so that for all $d \geq 0$,

$$M \odot A_d = \lambda(d)A_d,$$

which makes A_d look a lot like an eigenvector! And I guess it is: We can think of $\mathbb{Z}[G]$ as a $Sym_n(\mathbb{Z})$-module where a matrix $M \in Sym_n(\mathbb{Z})$ acts on an element $S \in \mathbb{Z}[G]$ by $M \odot S$. Then we have

Theorem 4 *Considering $\mathbb{Z}[G]$ as a $Sym_n(\mathbb{Z})$-module with Hadamard multiplication, then*

$$G \text{ is represented by } M \;\Leftrightarrow\; A_d \text{ is an eigenvector for } M \text{ for all } d$$

And then we can interpret the weights as the eigenvalues.

Now after this very interesting path, does it gain us any understanding for $Class(J_\lambda)$? Well we know G is represented by J_λ if and only if $J_\lambda \in Sym_n(\mathbb{Z})$ has A_d for its eigenvectors. So studying which graphs G are represented by some J_λ comes down to looking at its eigenvectors as a transformation $\mathbb{Z}[G] \to \mathbb{Z}[G]$.

But really, if A_d is an eigenvector from $\mathbb{Z}[G]$, then it will be so in $Sym_n(\mathbb{Z})$ as well. So we may as well consider the action of left multiplication on $Sym_n(\mathbb{Z})$ by itself, and the eigenvectors of J_λ under this action will tell us which graphs can be represented by it!

Let's make one more definition and then try to state this succinctly.

Definition 4 We'll call the collection $\{A_0, A_1, \ldots, A_{diam(G)}\}$ the *Adjacency Set* of G, denoted $Ad(G)$.

Then we get the following theorem:

Theorem 5 *The graph G is represented by M if and only if*

$$Ad(G) \subseteq EigenSpace(M)$$

Let's keep rolling - to find eigenvectors, it's usually easiest to find eigenvalues first and then solve for the eigenvectors. To do so, we need a characteristic polynomial, usually defined as

$$\chi_M(x) = det(Ix - M)$$

But we must remember that our multiplication is the Hadamard product. So I isn't the identity. Then we really want to solve

$$M \odot S = \lambda S$$

which means

$$M \odot S - \lambda S = 0$$

so

$$(M - \lambda J) \odot S = 0$$

Going to the determinant comes from the usual product, since

$$\exists \text{ non-trivial } S \textit{ such that } MS = 0 \iff det(M) = 0$$

So we need to see what an equivalent condition for the Hadamard product is. Suppose we had some non-trivial S so that $M \odot S = 0$. Then for all (i, j), we need either $M[i, j] = 0$ or $S[i, j] = 0$.

So the first fact we get is that if M contains any entries equal to 0, then it will have some non-trivial S so that $M \odot S = 0$. If no entries of M are 0, then $M \odot S = 0$ implies $S[i, j] = 0$ for all i, j, so $S = 0$, which means it isn't non-trivial! So for the Hadamard product, we get

$$\exists \text{ non-trivial } S \textit{ such that } M \odot S = 0 \iff M[i, j] = 0 \ for\ some\ (i, j)$$

So if $(M - \lambda J) \odot S = 0$, we require some entry of $M - \lambda J$ to be 0.

So for all entries $M[i, j]$, we get an eigenvalue $\lambda = M[i, j]$. And the corresponding eigenvector is the matrix with a single 1 where that entry appears (and its reflection since these are symmetric matrices), and 0 elsewhere. That's pretty cool! So letting $c_m(M) = \{(i, j) \mid i \geq j \textit{ and } M[i, j] = m\}$, this means

$$dim(EigenSpace_m(M)) = c_m(M).$$

It's maybe more helpful to think of a *maximal eigenvector* $Max_m(M)$ for m, which is the sum of all the generators of the eigenspace. Note that

$$J = \sum_{\substack{m \\ eigenvalue}} Max_m(M)$$

which is reminiscent of

$$J = \sum_{d=0}^{diam(G)} A_d$$

Again using $\leq$ for matrices to mean component-wise, we get

$$G\ is\ represented\ by\ M \iff \forall d \geq 0,\ A_d \leq Max_m(M),\ for\ some\ m$$

Distinguishing this as m_d, then $\lambda(d) = m_d$ gives the weights:

$$\sum_{d=0}^{diam(G)} m_d A_d = \sum_{\substack{m \\ eigenvalue}} m Max_m(M) = M$$

So what if we're asking for a dynamic M, i.e. $M = A^2(G)$, which depends on the graph. We've seen G is represented by $A^2(G)$ if and only if $\forall d \geq 0, A_d \leq Max_m(A^2(G))$, for some m.

I love all the work with the Hadamard product, but this does feel like it boils down to what we already knew about checking whether a matrix represents a graph or not: Simply take $M \odot A_d$ for each d and see if it's an eigenvector. If we succeed for all d, then M represents G. Otherwise, it doesn't.

4 Actually actually back to $\mathcal{S}_g$

Let's see if this one is successful! I want to address a few questions:

Question 5
1. *Is the chromatic number of $\mathcal{S}_g$ always g?*
2. *Does the chromatic polynomial of $\mathcal{S}_g$ always have its mode on x^{g-1}?*
3. *Is the critical group of $\mathcal{S}_g$ always cyclic?*
4. *Is the automorphism group of $\mathcal{S}_g$ always small?*
5. *Do any properties of $H_2(g)$ help us with $\mathcal{S}_g$?*

4.1 Chromatic Number

We have a few ways to bound $\chi(\mathcal{S}_g)$. Letting $\alpha(\mathcal{S}_g)$ be the independence number and $\omega(\mathcal{S}_g)$ be the clique number. Let $\Delta(\mathcal{S}_g)$ be the maximum degree. Then

$$\chi(\mathcal{S}_g) \geq \omega(\mathcal{S}_g)$$

$$\chi(\mathcal{S}_g) \geq \frac{N(g)}{\alpha(\mathcal{S}_g)}$$

$$\chi(\mathcal{S}_g) \leq \Delta(\mathcal{S}_g)$$

g	$\chi(\mathcal{S}_g)$	$\alpha(\mathcal{S}_g)$	$\omega(\mathcal{S}_g)$	$\Delta(\mathcal{S}_g)$
2	2	1	2	1
3	3	2	3	3
4	4	3	4	5
5	5	4	5	9
6	6	7	6	13
7	7	11	7	18
8	8	17	8	24

We definitely need some more data, but one thing seems clear, the chromatic number and clique number both seem to be the identity function. And now that I think of it, that makes sense, we just go to the trusty ordinary numerical semigroup $\mathcal{O}_g = \{1, 2, 3, \ldots, g\}^c$. It, along with the sets

$$\{1, 2, \ldots, g-1, g+1\}, \{1, 2, \ldots, g-1, g+2\}, \ldots, \{1, 2, \ldots, g-1, 2g-1\},$$

form a K_g.

And it shouldn't be hard to show this is maximum. If we assume $\chi(\mathcal{S}_g) = g$, then we get from the second equation

$$N(g) \leq g\alpha(\mathcal{S}_g)$$

The value of $g\alpha(\mathcal{S}_g)$ for $g = 2, 3, 4, 5, 6, 7, 8$ is

$$2, 6, 12, 20, 42, 77, 136$$

Compared to $N(g+1)$:

$$4, 7, 12, 23, 39, 67, 118$$

Compared to $N(g+2)$:

$$7, 12, 23, 39, 67, 118, 204$$

I wonder if we can interpret the independence number in a way to bound $g\alpha(\mathcal{S}_g)$ above by $N(g+2)$.

Conjecture 5 *For all g,*

$$N(g+2) \geq g\alpha(\mathcal{S}_g)$$

This would imply $N(g) \leq N(g+2)$ for all g.

Solving for the independence number, we'd need to show

$$\alpha(\mathcal{S}_g) \leq \frac{N(g+2)}{g}$$

Combinatorially, we'd want to find some way to build numerical semigroups in $\mathcal{N}_{g+2}$ using a maximum independent set in $\mathcal{S}_g$, which is just a collection of numerical semigroups of genus g.

The fact that they form an independent set I tells us that for any two $S_1, S_2 \in I$, we have $|S_1 \triangle S_2| > 2$. Write $S_1 \triangle S_2 = \{a_1, \ldots, a_k, b_1, \ldots, b_k\}$ where $a_i \in S_1^c$ and $b_i \in S_2^c$. Here is a maximum independent set in $\mathcal{S}_4$.

$$[\{1, 2, 3, 6\}, \{1, 2, 4, 7\}, \{1, 3, 5, 7\}]$$

Then

$$\{1, 2, 3, 6\} \triangle \{1, 2, 4, 7\} = \{3, 4, 6, 7\}$$
$$\{1, 2, 3, 6\} \triangle \{1, 3, 5, 7\} = \{2, 5, 6, 7\}$$
$$\{1, 2, 4, 7\} \triangle \{1, 3, 5, 7\} = \{2, 3, 4, 5\}$$

Well what if we add the two elements that aren't in the semigroup?

$$\{1, 2, 3, 6\}/\{1, 2, 4, 7\} \longrightarrow \{1, 2, 3, 4, 6, 7\}$$
$$\{1, 2, 3, 6\}/\{1, 2, 4, 7\} \longrightarrow \{1, 2, 3, 5, 6, 7\}$$
$$\{1, 2, 4, 7\}/\{1, 3, 5, 7\} \longrightarrow \{1, 2, 3, 4, 5, 7\}$$

5 Shifting Semigroups

If you were asking yourself, "can he define another graph?", then the answer is yes! Let's define the shift of a numerical semigroup G to be

$$\phi(S) = (S+1) \cup \{0\} - \{1\}.$$

Then $\phi(S)$ is often a numerical semigroup of genus $g+1$. Quantifying how often it fails could be another way to get a grasp on $N(g)$ and $N(g+1)$. As in, if $F(g)$ was the number of $S \in \mathcal{N}_g$ so that $\phi(S) \notin \mathcal{N}_{g+1}$, then

$$N(g+1) \geq N(g) - F(g)$$

As discussed in the Kaplan survey, Ye used the notion of a *strongly descended numerical semigroup* to prove that

$$N(g+1) \geq N(g) - N(g-1)$$

So the above is in a similar spirit, and if we could show $F(g) < N(g-1)$, it would be an improvement! Here are the values of $F(g)$ for $g = 1, 2, \ldots$:

$$0, 0, 1, 2, 3, 10, 15$$

This is not in the OEIS. And for comparison to $N(g-1)$:

$$0, 1, 2, 4, 7, 12, 23$$

So it seems reasonable! Could we use a failure for $\phi(S) \in \mathcal{N}_{g+1}$ to produce a numerical semigroup in $\mathcal{N}_{g-1}$? The first failure we encounter is

$$\phi(\{1, 3, 5, 7\}) = \{1, 2, 4, 6, 8\}$$

The fact that this fails comes from the fact that $6, 8 \in \{1, 2, 4, 6, 8\}$, while 3 and 5 are not. Removing these two gaps leads us with $\{1, 2, 4\}$, which is a numerical semigroup...does it generalize?

Our next failure is

$$\phi(\{1, 2, 3, 5, 9\}) = \{1, 2, 3, 4, 6, 10\},$$

which fails because of the 10. Removing it gives a numerical semigroup of genus g, so we could technically stop, but we could also remove the 6 and get a numerical semigroup $\{1, 2, 3, 4\}$.

Not sure that's the path, it's probably not wise to expect such a simple removal scheme to work. Instead, let's look at ϕ in a dynamic way and see what repeated applications does. For example, taking the first failure,

$$\phi^2(\{1, 3, 5, 7\}) = \phi(\{1, 2, 4, 6, 8\}) = \{1, 2, 3, 5, 7, 9\},$$

so $\phi^2(\{1, 3, 5, 7\}) \in \mathcal{N}_{g+2}$. Let's make a few definitions that will immediately lead to a few questions.

Definition 5 For a numerical semigroup S, we'll define the *shift index* of S to be the smallest positive integer k so that $\phi^k(S)$ is a numerical semigroup. And the *shift spectrum* of S will be the set of positive integers k so that $\phi^k(S)$ is a numerical semigroup.

Question 6 *1. Is the shift index always finite?*

2. Is the shift spectrum finite or infinite?

3. Does the shift spectrum contain any arithmetic patterns?

4. What is the natural density of the shift spectrum of a numerical semigroup?

Let's first observe that the ordinary semigroup $\mathcal{O}_g = \{1, 2, \ldots, g\}^c$ is a fixed point of sorts:

$$\phi(\mathcal{O}_g) = \mathcal{O}_{g+1}$$

What if we iterate for a different numerical semigroup? For example,

$$\{1,3,5,7\} \longrightarrow \{1,2,4,6,8\} \longrightarrow \{1,2,3,5,7,9\} \longrightarrow \{1,2,3,4,6,8,10\} \longrightarrow$$

$$\longrightarrow \{1,2,3,4,5,7,9,11\} \longrightarrow \{1,2,3,4,5,6,8,10,12\} \longrightarrow \{1,2,3,4,5,6,7,9,11,13\}$$

So we actually stabilize on numerical semigroups. So the shift spectrum of $\{1, 3, 5, 7\}$ is $\{0, 2, 4, 5, 6, \ldots\}$, which is itself a numerical semigroup. That's fun, but I can't imagine generalizes.

The first observation shows that for all g, the shift spectrum of $\mathcal{O}_g$ is $\{0, 1, 2, 3, 4, \ldots\}$. I believe we can also say that the shift spectrum will eventually contain every integer larger than some N, because of how the multiplicity and Frobenius number interact. Note

$$m(\phi(S)) = m(S) + 1$$

$$F(\phi(S)) = F(S) + 1$$

Iterating this gives

$$m(\phi^k(S)) = m(S) + k$$
$$F(\phi^k(S)) = F(S) + k$$

so $F(\phi^k(S)) - 2m(\phi^k(S)) = F(S) - 2m(S) - k$. The following proposition shows that this mean $\phi^k(S)$ is a numerical semigroup.

Proposition 3 *If $max(S^c) < 2min(S - \{0\})$, then S is a numerical semigroup.*

Proof 2 *First, by definition, the smallest non-zero element of S is $m(S)$, so the smallest failure would be if $m(S) + m(S) = 2m(S)$. By definition, $F(S)$ is the largest element not in S. Therefore, if $F(S) < 2m(S)$, we can't have any failures. So S is closed under addition and is therefore a numerical semigroup.*

I also want to say that Zhai, Kaplan, and Ye's work all deals with separating semigroups based off $F(S) < 2m(S)$. And it's not hard to state the amazing result: The number of numerical semigroups of genus g satisfying $F(S) < 2m(S)$ is the $g + 1^{st}$ Fibonacci number.

So the proposition isn't new but it is helpful! And this was a fun way to get to numerical semigroups satisfying $F(S) < 2m(S)$. Let's continue!

Since $F(\phi^k(S)) - 2m(\phi^k(S)) = F(S) - 2m(S) - k$, we know it will be less than 0 for $k > F(S) - 2m(S)$. And so...

Theorem 6 *The shift operator will eventually stabilize on numerical semigroups for any numerical semigroup S. Specifically, the shift spectrum of S will contain all integers larger than $F(S) - 2m(S)$.*

Let $\psi(g, k)$ be the number of numerical semigroups of genus g with shift index k.

Corollary 3 *Any numerical semigroup with $F(S) < 2m(S)$ will have shift index 1. By Zhao's result,*

$$\psi(g, 1) \geq Fib(g + 1)$$

Otherwise, the shift index is at most $F(S) - 2m(S) + 1$.

Of course, it would be awesome to see if this shift idea leads to a different proof of that result. Also note the "at most". The shift index seems to often be less than this value.

But I also think sometimes it's important to recognize goals you've accomplished. And I wanted to celebrate for a moment that we've answered all four questions in Question 6! Except maybe 3, depending on how we define "arithmetic patterns".

Like the other stats, we have

$$\sum_{k=1}^{\infty} \psi(g, k) = N(g)$$

And this means

$$N(g) - N(g-1) = \sum_{k=1}^{\infty} \psi(g, k) - \psi(g-1, k)$$

so we arrive again at the whole point of these statistics - if $\psi(g, k) \geq \psi(g-1, k)$ for all k, then this proves the desired $N(g) \geq N(g-1)$. But remember for the last two attempts with this kind of statement, it turned out not to hold for each k. Let's see if this one does!

We'll define the statistic generating function as

$$\Psi(g) = \sum_{k=1}^{\infty} x^{\psi(g,k)}$$

Here are the first few:

$$\Psi(1) = x$$

$$\Psi(2) = 2x$$

$$\Psi(3) = x^2 + 3x$$

$$\Psi(4) = 2x^2 + 5x$$

$$\Psi(5) = 3x^2 + 9x$$

$$\Psi(6) = x^4 + 2x^3 + 7x^2 + 13x$$

The one numerical semigroup of genus 6 with shift index 4 is:

$$\{1,2,3,6,7,11\} \to \{1,2,3,4,7,8,12\} \to \{1,2,3,4,5,8,9,13\} \to$$

$$\to \{1,2,3,4,5,6,9,10,14\} \to \{1,2,3,4,5,6,7,10,11,15\}$$

To get a better feel for this (as the coefficients seem to be increasing for a fixed exponent), let's look at some shift spectra. We'll only list those semigroups with $F(S) > 2m(S)$.

$$\begin{aligned} NumSgp &: Spectra \\ \{1,3,5\} &: \{0,2,3,\dots\} \\ \{1,2,4,7\} &: \{0,2,3,\dots\} \\ \{1,3,5,7\} &: \{0,2,4,5,\dots\} \end{aligned}$$

I think the best way to visualize this is by writing out the chains formed and coloring a set blue if it's a numerical semigroup and red otherwise. We'll color the starts of the chains with magenta. You may have to zoom in.

$$\begin{tikzcd}
\{1\} \arrow[r] & \{1,2\} \arrow[r] & \{1,2,3\} \arrow[r] & \{1,2,3,4\} \arrow[r] & \{1,2,3,4,5\} \arrow[r] & \{1,2,3,4,5,6\} \\
 & \{1,3\} \arrow[r] & \{1,2,4\} \arrow[r] & \{1,2,3,5\} \arrow[r] & \{1,2,3,4,6\} \arrow[r] & \{1,2,3,4,5,7\} \\
 & & \{1,2,5\} \arrow[r] & \{1,2,3,6\} \arrow[r] & \{1,2,3,4,7\} \arrow[r] & \{1,2,3,4,5,8\} \\
 & & \{1,3,5\} \arrow[r] & \{1,2,4,6\} \arrow[r] & \{1,2,3,5,7\} \arrow[r] & \{1,2,3,4,6,8\} \\
 & & & \{1,2,3,7\} \arrow[r] & \{1,2,3,4,8\} \arrow[r] & \{1,2,3,4,5,9\} \\
 & & & \{1,2,4,5\} \arrow[r] & \{1,2,3,5,6\} \arrow[r] & \{1,2,3,4,6,7\} \\
 & & & \{1,2,4,7\} \arrow[r] & \{1,2,3,5,8\} \arrow[r] & \{1,2,3,4,6,9\} \\
 & & & \{1,3,5,7\} \arrow[r] & \{1,2,4,6,8\} \arrow[r] & \{1,2,3,5,7,9\} \\
 & & & & \{1,2,3,4,9\} \arrow[r] & \{1,2,3,4,5,10\} \\
 & & & & \{1,2,3,5,9\} \arrow[r] & \{1,2,3,4,6,10\} \\
 & & & & \{1,2,3,6,7\} \arrow[r] & \{1,2,3,4,7,8\} \\
 & & & & \{1,2,4,5,7\} \arrow[r] & \{1,2,3,5,6,8\} \\
 & & & & \{1,2,4,5,8\} \arrow[r] & \{1,2,3,5,6,9\} \\
 & & & & \{1,3,5,7,9\} \arrow[r] & \{1,2,4,6,8,10\}
\end{tikzcd}$$

If $\mathcal{N}_\infty$ denotes the set of all numerical semigroups, then this is just the closure under shifting, call it $\overline{\mathcal{N}}_\infty$. Let's denote by $\overline{N}(g)$ the number of elements in the g^{th} column of the above figure. The values $\overline{N}(g)$ for $g = 1, 2, \dots$ are

$$1, 2, 4, 8, 14, 26, 49, 82$$

It's not in the OEIS, but a close sequence might be of interest: A164167

One thing that looks very interesting is the $\mathcal{N}_g$ seems to be split completely evenly if $N(g)$ is even and $(N(g) - 1)/2$ blue and $(N(g) + 1)/2$ magenta, if $N(g)$ is odd. If we could prove that, then it would lead to a very interesting simplification: We would only need to prove either the number of blue elements are increasing or the number of magenta elements are. Then we'd get the whole is increasing.

Let's call the magenta elements the *unshifted numerical semigroups* of genus g, denoted $\mathcal{N}_0(g)$, and the number of such semigroups $N_0(g)$.

Now what is all this giving us? Well, first of all, it's clear that

$$\overline{N}(g) \geq \overline{N}(g-1)$$

for all g, since shifting each element in one column gives a unique element in the next column. We can be more specific and say

$$\overline{N}(g) = \overline{N}(g-1) + N_0(g)$$

which inductively gives

$$\overline{N}(g) = \sum_{k=1}^{g} N_0(k)$$

Conjecture 6

$$N_0(g) = \left\lceil \frac{N(g)}{2} \right\rceil$$

Let's go back to that OEIS sequence A164167. It gives the number of binary strings of length n with equal numbers of 0010 and 0101 substrings:

$$1, 2, 4, 8, 14, 26, 49, 92$$

So it agrees with $\overline{N}(g)$ except for 92, for which $\overline{N}(g)$ is 82. The entry links to a paper solving a problem Richard Stanley proposed asking for the number of strings of length n on the alphabet $\{H, T\}$ so that there are as many occurrences of HT as there are TT. But I'll stick with the binary string perspective.

So the $1, 2, 4, 8$ just comes from all binary strings of length $0, 1, 2, 3$, but then the 14 comes from $16 - 2$, because we can't have the strings 0010 or 0101. Note the offset: $\overline{N}(g) \approx B(g-1)$, where $B(n)$ is the OEIS sequence.

So then the question is: Can we take every element in $\overline{\mathcal{N}}_g$ and produce a binary string of length $g-1$ that avoids the two patterns 0010 and 0101? I think the place to start would be examining what two sets are "missing" from $\overline{\mathcal{N}}_5$

It *feels* natural that $0000 \leftrightarrow \{1, 2, 3, 4, 5\}$ and $1111 \leftrightarrow \{1, 3, 5, 7, 9\}$, or vice versa. It reminds me of the amazing guidance of Bruce Sagan during an REU at Michigan State University after my freshman year at JMU. He showed me what math research was really like and introduced me to the beautiful world of combinatorics. And I think I might have found the presentation he gave us to pitch his idea at the start of the REU! The date matchs up, I think. And I remember I went with the idea of doing another project, but when I saw his presentation, I was so interested I couldn't do anything else!

One thing that came from that is the use of $0 - hat$ ($\hat{0}$) and $1 - hat$ ($\hat{1}$) to denote the maximum element and minimum element of a poset. The two numerical semigroups $\{1, 2, \ldots, g\}^c$ and $\{1, 3, \ldots, 2g-1\}^c$ definitely feel like a $\hat{0}$ and $\hat{1}$ to me! And the fact that we have the offset $\overline{N}(g) \approx B(g-1)$ actually makes the idea very clear: Look at the gaps! We'll put a 0 if there is no gap and a 1 if there is

$$\{1,2,3\} \leftrightarrow 00$$
$$\{1,2,4\} \leftrightarrow 01$$
$$\{1,2,5\} \leftrightarrow 01$$
$$\{1,3,5\} \leftrightarrow 11$$

So while doing so did get the $\hat{0}$ and $\hat{1}$ that we thought we should, both middle sets map to 01. But they are different colors, so maybe that's the fix...Let's see what we get with genus 4.

$$\begin{array}{ll} \{1,2,3,4\} \leftrightarrow 000 & \{1,2,4,5\} \leftrightarrow 010 \\ \{1,2,3,5\} \leftrightarrow 001 & \{1,2,4,6\} \leftrightarrow 011 \\ \{1,2,3,6\} \leftrightarrow 001 & \{1,2,4,7\} \leftrightarrow 011 \\ \{1,2,3,7\} \leftrightarrow 001 & \{1,3,5,7\} \leftrightarrow 111 \end{array}$$

One issue with this strategy is clear: Such strings will never start with a 1 unless it's from $\{1, 3, \ldots, 2g-1\}$. These won't be binary strings, but let's go ahead and write out the differences (-1) between consecutive gaps:

$$\begin{array}{ll} \{1,2,3,4\} \leftrightarrow 000 & \{1,2,4,5\} \leftrightarrow 010 \\ \{1,2,3,5\} \leftrightarrow 001 & \{1,2,4,6\} \leftrightarrow 011 \\ \{1,2,3,6\} \leftrightarrow 002 & \{1,2,4,7\} \leftrightarrow 012 \\ \{1,2,3,7\} \leftrightarrow 003 & \{1,3,5,7\} \leftrightarrow 111 \end{array}$$

I think I see a pattern: When the first two spots are 0, the last spot can be $0, 1, 2, 3$. If the second spot is a 1, the last spot can be $0, 1, 2$, and if both first spots are 1, then the last spot has to be a 1.

In fact, we know for all of these strings that if the first spot is a 1, the rest must be 1s. The binary strings we picked up in this way are $000, 001, 010, 011, 111$, so can we do something to turn

$$002, 003, 012 \longrightarrow 100, 101, 110$$

Maybe removing a 2 at the end adds a 1 at the beginning?

$$002 \longleftrightarrow 100 \qquad 003 \longleftrightarrow 101 \qquad 012 \longleftrightarrow 110$$

Notice that this does fix the issue with genus 5 as well, since $\{1, 2, 5\} \leftrightarrow 02 \leftrightarrow 10$. Let's see what happens when we move to genus 5. We'll do the same thing by taking consecutive differences and shifting any entry larger than a 1 to the largest spot that is 0.

$\{1,2,3,4,5\} \leftrightarrow 0000 \leftrightarrow 0000$	$\{1,2,3,5,8\} \leftrightarrow 0012 \leftrightarrow 1010$
$\{1,2,3,4,6\} \leftrightarrow 0001 \leftrightarrow 0001$	$\{1,2,3,5,9\} \leftrightarrow 0013 \leftrightarrow 1011$
$\{1,2,3,4,7\} \leftrightarrow 0002 \leftrightarrow 1000$	$\{1,2,3,6,7\} \leftrightarrow 0020 \leftrightarrow 1000$
$\{1,2,3,4,8\} \leftrightarrow 0003 \leftrightarrow 1001$	$\{1,2,4,5,7\} \leftrightarrow 0101 \leftrightarrow 0101$
$\{1,2,3,4,9\} \leftrightarrow 0004 \leftrightarrow 1100$	$\{1,2,4,5,8\} \leftrightarrow 0102 \leftrightarrow 1100$
$\{1,2,3,5,6\} \leftrightarrow 0010 \leftrightarrow 0010$	$\{1,2,4,6,8\} \leftrightarrow 0111 \leftrightarrow 0111$
$\{1,2,3,5,7\} \leftrightarrow 0011 \leftrightarrow 0011$	$\{1,3,5,7,9\} \leftrightarrow 1111 \leftrightarrow 1111$

So we see some more repeats with the method. We're missing

$$0110, 0100, 1000, 1000, 1001, 1010, 1011, 1100, 1101$$

The two repeats are $0004 \leftrightarrow 1100$ and $0102 \leftrightarrow 1100$. This can be fixed by sending 0102 to 0110 instead. Maybe any 1 acts like a "wall".

If we do this, then the two binary strings of length 4 that we're missing are

$$0100,\ 1010$$

which are exactly the flips of the two strings OEIS gave! Let's be explicit about the proposed injection:

- Given S^c gapset of a numerical semigroup, compute

  $$Dif(S) = \{a_{i+1} - a_i - 1 \mid 0 \le i \le g-1\},$$

  and write it as a string.

- Starting at the right and moving left, if an entry is larger than 1, then subtract 2 and place a 1 as far left as you can without crossing over any non-zero entries.

- If you encounter a substring $\dots 112 \dots$, then subtract 2 and place a 1 in the same way as before, but skipping over any initial sequence of non-zero numbers.

- Call this final string $\overline{Dif}(S)$.

Having larger numbers on the right forces smaller numbers on the left, since you can only have a large separation at the end if you have many consecutive or near-consecutive gaps at the beginning. Further, the separation can never be larger than $\{1, 2, \dots, g-1, 2g-1\}$, for which $Dif = \{0, 0, \dots, 0, g-1\}$. Because of that, it seems possible to me that we can always perform this process! Where does the extremal set go?

$$\{1, 2, \dots, g-1, 2g-1\} \longleftrightarrow 0000\dots0g-1 \longleftrightarrow 1000\dots0g-3 \longleftrightarrow$$

$$\longleftrightarrow 1100\dots0g-5 \longleftrightarrow \dots$$

In the end, we'll have either $11\dots100\dots00$ or $11\dots100\dots01$, depending on the parity of g.

Let's state the conjecture we are working on, to be complete:

Conjecture 7 *For all g,*

$$\overline{N}(g) \leq B(g-1)$$

where $B(n)$ is the number of binary strings of length n containing an equal number of 0100 *and* 1010 *substrings.*

What do those substrings mean for the numerical semigroup?

$$0100 \longleftrightarrow x, x+1, x+3, x+4, x+5$$

$$1010 \longleftrightarrow 0012 \longleftrightarrow x, x+1, x+2, x+4, x+7$$

We definitely need to look at some larger genus semigroups to get a grasp on this. Maybe a higher genus numerical semigroup will allow us to see "paired" 0100 and 1010 strings.

$$S = \{1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 15, 16, 18, 19\}^c$$

$$Dif(S) = 0000010102010$$

$$\overline{Dif}(S) = 0000010110010$$

No such substrings. Let's try to force 0100.

$$S = \{1, 2, 3, 4, 5, 6, 8, 9, 10\}^c$$

$$Dif(S) = 00000100$$

And another example

$$S = \{1, 2, 3, 4, 5, 7, 8, 10, 11\}$$

$$Dif(S) = 00001010$$

$$S = \{1, 2, 3, 4, 5, 7, 8, 10, 13\}$$

$$Dif(S) = 00001012$$

$$\overline{Dif}(S) = 00001110$$

Maybe the OEIS sequence is a bit of a red herring...what can we say independently about the strings we form using the proposed injection?

Is it even an injection? For example, the string 110 could arise by itself or as 012, as evidenced by the last example. Or it could mean the map isn't right. Maybe we should just treat $Dif(S)$ as a binary number and convert it by carrying like normal. Let's see

$$\{1,2,3,4\} \leftrightarrow 000 \leftrightarrow 000 \qquad \{1,2,4,5\} \leftrightarrow 010 \leftrightarrow 010$$

$$\{1,2,3,5\} \leftrightarrow 001 \leftrightarrow 001 \qquad \{1,2,4,6\} \leftrightarrow 011 \leftrightarrow 011$$
$$\{1,2,3,6\} \leftrightarrow 002 \leftrightarrow 010 \qquad \{1,2,4,7\} \leftrightarrow 012 \leftrightarrow 100$$
$$\{1,2,3,7\} \leftrightarrow 003 \leftrightarrow 011 \qquad \{1,3,5,7\} \leftrightarrow 111 \leftrightarrow 111$$

Still repeats.

Let's think about it in another way. The semigroup $\mathcal{O}_g = \hat{0}$ is never unshifted except for $g = 1$, while the numerical semigroup $\hat{1} = \{1, 3, 5, \ldots, 2g-1\}$ is always unshifted. If we can figure out a way to get $\hat{1}$ and $\hat{0}$ to swap, maybe we can extend it to the rest to prove $N_0(g) = \lceil g/2 \rceil$.

Unshifted semigroups often have higher Frobenius numbers, but shifted semigroups can also. But I guess since the maximum of $F(S)$ for a numerical semigroup of genus g is $2g - 1$, we know that if $F(S') = 2g + 1$, then we could not have shifted to it. So we get

$$N_0(g) \geq |\{S \in \mathcal{N}_g \mid F(S) = 2g - 1\}|$$

The number of numerical semigroups with $F(S) = 2g - 1$ for $g = 1, 2, \ldots$ is

$$1, 1, 2, 3, 3, 7, 8, 7$$

This does not appear in the OEIS, which is honestly very surprising since it seems like a natural thing to look at for numerical semigroups!

It's because I had a mistake in $\mathcal{N}_6$. Crap! I've fixed the list now, but the change here is that the 7 is supposed to be a 6. With this fix, we do indeed find an OEIS sequence A158278, counting the number of such semigroups, called *Symmetric Numerical Semigroups*. But we'll touch more on that later.

For example, if $f(S)$ is any function like Frobenius number, multiplicity. And if we're crazy enough, we could encode this all in a multi-variate generating function!

$$S_g(x, y) = \sum_{S \in \mathcal{N}_g} x^{F(S)} y^{m(S)}$$

Here they are for $g = 1, 2, \ldots$

$$S_1(x, y) = xy^2$$

$$S_2(x, y) = x^3y^2 + x^2y^3 = x^2y^2(x + y)$$
$$S_3(x, y) = x^5y^3 + x^5y^2 + x^4y^3 + x^3y^4 = x^3y^2(x^2y + x^2 + xy + y^2)$$
$$S_4(x, y) = x^7y^4 + x^7y^3 + x^6y^4 + x^7y^2 + x^5y^4 + x^4y^5 + x^5y^3$$

$$= x^4y^2(x^3y^2 + x^3y + x^2y^2 + x^3 + xy^2 + y^3 + xy)$$

$$S_5(x,y) = x^5y^2(x^4y^3 + x^4y^2 + x^3y^3 + x^2y^3 + x^4 + x^3y + 2x^2y^2 + xy^3 + y^4 + x^2y + xy^2)$$

Notice we get our first non-1 coefficient for x^7y^4, coming from the semigroups

$$\{1,2,3,5,7\}$$

$$\{1,2,3,6,7\}$$

Fixing a genus g, multiplicity m, and Frobenius number F, we have a restricted number of choices:

$$\{1,2,\ldots,m-1,\ldots,F\}$$

The red dots are undetermined. As we fixed the genus, we need to choose $g-m$ integers between $m+1$ and F so that we still end up with a numerical semigroup. In total we have

$$\binom{F-m-1}{g-m}$$

possibilities, but most will probably not give a numerical semigroup.

I want to go back to looking at shift spectra. I'll list only the non-trivial ones, and I'll only list numbers before $F - 2m(S)$, since we know it contains every integer after that.

$$SSpec(\{1,3,5\}) = [0,2]$$

$$SSpec(\{1,2,4,7\}) = [0,2]$$

$$SSpec(\{1,3,5,7\}) = [0,2,4]$$

$$SSpec(\{1,2,3,5,9\}) = [0,2]$$

$$SSpec(\{1,2,4,5,7\}) = [0,2]$$

$$SSpec(\{1,2,4,5,8\}) = [0,3]$$

$$SSpec(\{1,3,5,7,9\}) = [0,2,4,6]$$

$$SSpec(\{1,2,3,4,6,11\}) = [0,2]$$

$$SSpec(\{1,2,3,4,8,11\}) = [0,2]$$

$$SSpec(\{1,2,3,5,6,9\}) = [0,2]$$

$$SSpec(\{1,2,3,5,6,10\}) = [0,1,3]$$

$$SSpec(\{1,2,3,5,7,9\}) = [0,2]$$

$$SSpec(\{1,2,3,5,7,11\}) = [0,2,4]$$

$$SSpec(\{1,2,3,6,7,11\}) = [0,4]$$

$$SSpec(\{1,2,4,5,7,8\}) = [0,3]$$

$$SSpec(\{1,2,4,5,7,10\}) = [0,2,3]$$

$$SSpec(\{1,2,4,5,8,11\}) = [0,3,4,6]$$

$$SSpec(\{1,3,5,7,9,11\}) = [0,4,6,8]$$

What we really want is a way to generate the unshifted semigroups of genus g using the shifted semigroups of genus g. This would constructively show that $N(g) \geq N(g-1)$.

All this thinking of binary strings makes me realize we haven't done the simplest string of all - the indicator function! Maybe looking at shifts in this way reveals some forbidden pattern. So we're interested in binary strings of length $2g-1$ that have exactly g 1s. The number of all such strings is

$$\binom{2g-1}{g}.$$

And if that looks familiar, it's because it's the number of vertices of $H_2(g)$ that we defined at the very beginning!

But that graph connected S and S' if $|S \triangle S'| = 2$. Now we want to take the binary strings corresponding to sets in $\overline{\mathcal{N}}_g$ and see how they respond to shifting. We'll rewrite the large picture we had before but with binary strings:

$$\begin{tikzcd}
1 \arrow[r] & 110 \arrow[r] & 11100 \arrow[r] & 1111000 \arrow[r] & 111110000 \arrow[r] & 11111100000 \\
 & 101 \arrow[r] & 11010 \arrow[r] & 1110100 \arrow[r] & 111101000 \arrow[r] & 11111010000 \\
 & & 11001 \arrow[r] & 1110010 \arrow[r] & 111100100 \arrow[r] & 11111001000 \\
 & & 10101 \arrow[r] & 1101010 \arrow[r] & 111010100 \arrow[r] & 11110101000 \\
 & & & 1110001 \arrow[r] & 111100010 \arrow[r] & 11111000100 \\
 & & & 1101100 \arrow[r] & 111011000 \arrow[r] & 11110110000 \\
 & & & 1101001 \arrow[r] & 111010010 \arrow[r] & 11110100100 \\
 & & & 1010101 \arrow[r] & 110101010 \arrow[r] & 11101010100 \\
 & & & & 111100001 \arrow[r] & 11111000010 \\
 & & & & 111010001 \arrow[r] & 11110100010 \\
 & & & & 111001100 \arrow[r] & 11110011000 \\
 & & & & 110110100 \arrow[r] & 11101101000 \\
 & & & & 110110010 \arrow[r] & 11101100100 \\
 & & & & 101010101 \arrow[r] & 11010101010
\end{tikzcd}$$

The first thing we can notice is that all blue strings end in a zero, which is saying the same thing we did before: If S has $F(S) = 2g-1$, then it's unshifted.

The next thing we can notice is that for $g \geq 4$, the blue strings always have at least 111 at the beginning. This is because to get to 110, we would have needed to come from 10, which means the semigroup starts $\{1, 3, \ldots\}$, but that means it must be $\{1, 3, \ldots, 2g-1\}$, which shifts to $\{1, 2, 4, 6, \ldots, 2g\}$, which, isn't a numerical semigroup for $g \geq 3$.

Let's also explicitly mention the shift: In this point of view, a shift is just adding a 1 to the beginning of the string. As such, maybe removing the 1 from the front and moving it *somewhere* will produce an unshifted semigroup. Let's shift to the end:

For example,

$$1111000 \to 1110001$$

$$1110100 \to 1101001$$

$$1110010 \to 1100101$$

The first two are successes but not the last, which corresponds to $\{1, 2, 5, 7\}$.

$$111110000 \to 111100001$$

$$111101000 \to 111010001$$

$$111100100 \to 111001001$$

$$111010100 \to 110101001$$

$$111100010 \to 111000101$$

$$111011000 \to 110110001$$

And the reason why this doesn't work is obvious: It's attaching $2g - 1$ to every single semigroup, and not all unshifted sets actually have $F(S) = 2g - 1$. And on top of that, having $2g-1$ makes many of these sets not numerical semigroups because of the lack of closure of addition.

So here's a vague solution/question: Can we move the initial 1 anywhere in each shifted semigroup to produce all (except maybe 1) unshifted one, even ad hoc with each semigroup?

$$1111000 \to 1110001$$

$$1110100 \to 1101001$$

$$1110010 \to \ ?$$

Moving a single 1 in the last semigroup cannot produce either 1101100 or 1010101, which is what I expected. Moving a single 1 can produce

$$1111000 \leftrightarrow 1110001, 1101100, 1101001$$

$$1110100 \leftrightarrow 1110001, 1101100$$

$$1110010 \leftrightarrow 1110001$$

so the last one is forced, and then we have a choice for the first two. In other words, moving a single 1 does produce two injections from $\mathcal{N}_g \backslash \mathcal{N}_0(g)$ to $\mathcal{N}_0(g)$.

But I did just realize I think we're recreating $H_2(g)$ but restricted to $\overline{\mathcal{N}}(g)$ instead of $\mathcal{N}(g)$. Because "moving a single 1" literally means the two semigroups differ in a single element.

6 Cyclic Shifts

Ok, here's something I want to try with no expectations of showing anything: Another amazing combinatorial proof (that I can't specifically remember) was showing some value was less than $F(q)/q$ where $F(q)$ is some function of q, and they do so by showing a bijection and then showing that cyclic (mod q) shifts map to the same place, which explains the division by q.

Keeping this in mind, instead of shifting $\mathcal{N}_g \to \mathcal{N}_{g+1}$, maybe we can start with $\mathcal{N}_g$ and do cyclic shifts within itself, expanding the set as neccessary - in similar language to before, looking at the closure under cyclic shifts.

One thing this will obviously produce is strings starting in 0. Specifically, since we have $g-1$ 0s in each string, we'll have exactly $g-1$ strings that start with a zero for each semigroupng, we'll have exactly $g-1$ strings that start with a zero for each semigroup. We'll denote this as $Cyc(S)$ for a numerical semigroup S.

$$110 \to 011 \to 101 \to 110$$

$$\{1,2\} \to \{2,3\} \to \{1,3\} \to \{1,2\}$$

$$- - - - - - - - - - - - - - - -$$

$$11100 \to 01110 \to 00111 \to 10011 \to 11001 \to 11100$$

$$\{1,2,3\} \to \{2,3,4\} \to \{3,4,5\} \to \{1,4,5\} \to \{1,2,5\} \to \{1,2,3\}$$

$$- - - - - - - - - - - - - - - -$$

$$11010 \to 01101 \to 10110 \to 01011 \to 10101 \to 11010$$

$$\{1,2,4\} \to \{2,3,5\} \to \{1,3,4\} \to \{2,4,5\} \to \{1,3,5\}$$

A helpful idea here is "shape". If we throw the numbers $1, 2, \ldots, 2g-1$ on a circle, then fixing a particular "shape" and rotating it around the wheel gives us more sets, but they all belong to the same family.

So this associates to every numerical semigroup of genus g a collection of $2g-1$ sets. If it's helpful, we could only pick up the numerical semigroups from this. For example, the above shows that this partitions $\mathcal{N}_3$ into two equal sets:

$$\{\{1,2,3\},\{1,2,5\} \; and \; \{\{1,2,4\},\{1,3,5\}\}$$

which, not to get too excited, is one shifted semigroup and one unshifted semigroup in each part. Because we're doing this to study the shifting, we'll include all elements of $\overline{\mathcal{N}}_g$. And now that I think about it, if we restrict to the ones that have a 1 at the front, we'll get exactly g sets in each cycle. Let's do $g = 4$.

$$1111000 \to 1000111 \to 11000011 \to 1110001$$

$$\{1,2,3,4\}, \{1,5,6,7\}, \{1,2,6,7\}, \{1,2,3,7\}$$

$$1110100 \to 1001110 \to 1010011 \to 1101001$$

$$\{1,2,3,5\},\{1,4,5,6\},\{1,3,6,7\},\{1,2,4,7\}$$

$$1110010 \to 1011100 \to 1001011 \to 1100101$$

$$\{1,2,3,6\},\{1,3,4,5\},\{1,4,6,7\},\{1,2,5,7\}$$

$$1101010 \to 1011010 \to 1010110 \to 1010101$$

$$\{1,2,4,6\},\{1,3,4,6\},\{1,3,5,6\},\{1,3,5,7\}$$

And there's an unmatched string! We don't get $\{1,2,4,5\}$, so let's look at its cycle:

$$1101100 \to 1001101 \to 1100110 \to 1011001$$

$$\{1,2,4,5\},\{1,4,5,7\},\{1,2,5,6\},\{1,3,4,7\}$$

Wow, this is really cool, and possibly a great way to associate shifted sets (blue,red) with unshifted sets (magenta).

But we definitely need to go to a higher genus to get a feel for what each cycle actually looks like. Let's choose one at random. I'm highlighting the first 1 to keep track easier.

$$11110110101100000 \to 10000011110110101 \to 11000001111011010 \to$$

$$\to 10110000011110110 \to 10101100000111101 \to 11010110000011110 \to$$

$$\to 10110101100000111 \to 11011010110000011 \to 11101101011000001$$

And in sets:

$$\{1,2,3,4,6,7,9,11,12\} \to \{1,7,8,9,10,12,13,15,17\} \to \{1,2,8,9,10,11,,13,14,16\}$$

$$\to \{1,3,4,10,11,12,13,15,16\} \to \{1,3,5,6,12,13,14,15,17\} \to \{1,2,4,6,7,13,14,15,16\}$$

$$\to \{1,3,4,6,8,9,15,16,17\} \to \{1,2,4,5,7,9,10,16,17\} \to \{1,2,3,5,6,8,10,11,17\}$$

Let's do some "common" numerical semigroups. But before that, it'll be much easier to code this up first.

```
def cycle(b):
    cyc = [b[-1]]
    for i in range(len(b)-1):
        cyc.append(b[i])
    return cyc


def CyclicShift(S):
    bit = []
    g = len(S)
    for i in range(1,2*g):
        if i in S:
            bit.append(1)
        else:
```

```
            bit.append(0)
    Cycle = [S]
    sbit = bit
    for j in range(2*g-1):
        sbit = cycle(sbit)
        if sbit[0] == 1:
            sset = []
            for i in range(2*g-1):
                if sbit[i] == 1:
                    sset.append(i+1)
            if Set(sset) in V5:
                Cycle.append((Set(sset),'NS'))
            else:
                Cycle.append(Set(sset))
    return Cycle
```

where V5 is a set containing all numerical semigroups of genus 5, and similarly for Vg. Unfortunately, this won't catch the shifted non-numerical semigroups...for now I'll go ahead.

$$[\{1,2,3,4,5\},\{1,6,7,8,9\},\{1,2,7,8,9\},\{1,2,3,8,9\},\{1,2,3,4,9\}]$$
$$[\{1,2,3,4,6\},\{1,5,6,7,8\},\{1,3,7,8,9\},\{1,2,4,8,9\},\{1,2,3,5,9\}]$$
$$[\{1,2,3,4,7\},\{1,4,5,6,7\},\{1,4,7,8,9\},\{1,2,5,8,9\},\{1,2,3,6,9\}]$$
$$[\{1,2,3,4,8\},\{1,3,4,5,6\},\{1,5,7,8,9\},\{1,2,6,8,9\},\{1,2,3,7,9\}]$$
$$[\{1,2,3,5,6\},\{1,5,6,7,9\},\{1,2,6,7,8\},\{1,3,4,8,9\},\{1,2,4,5,9\}]$$
$$[\{1,2,3,5,7\},\{1,4,5,6,8\},\{1,3,6,7,8\},\{1,3,5,8,9\},\{1,2,4,6,9\}]$$
$$[\{1,2,3,6,7\},\{1,4,5,6,9\},\{1,2,5,6,7\},\{1,4,5,8,9\},\{1,2,5,6,9\}]$$
$$[\{1,2,4,5,7\},\{1,4,5,7,8\},\{1,3,6,7,9\},\{1,2,4,7,8\},\{1,3,4,6,9\}]$$
$$[\{1,2,4,5,8\},\{1,3,4,6,7\},\{1,4,6,7,9\},\{1,2,5,7,8\},\{1,3,4,7,9\}]$$
$$[\{1,3,5,7,9\},\{1,2,4,6,8\},\{1,3,4,6,8\},\{1,3,5,6,8\},\{1,3,5,7,8\}]$$
$$[\{1,2,3,5,8\},\{1,3,4,5,7\},\{1,4,6,7,8\},\{1,3,6,8,9\},\{1,2,4,7,9\}]$$

And for $g = 6$:

$$[\{1,2,3,4,5,6\},1,7,8,9,10,11,1,2,8,9,10,11,1,2,3,9,10,11,1,2,3,4,10,11,\{1,2,3,4,5,11\}]$$
$$[\{1,2,3,4,5,7\},1,6,7,8,9,10,1,3,8,9,10,11,1,2,4,9,10,11,1,2,3,5,10,11,\{1,2,3,4,6,11\}]$$
$$\{1,2,3,4,5,8\},\{1,5,6,7,8,9\},\{1,4,8,9,10,11\},\{1,2,5,9,10,11\},\{1,2,3,6,10,11\},\{1,2,3,4,7,11\}]$$
$$[\{1,2,3,4,5,9\},\{1,4,5,6,7,8\},\{1,5,8,9,10,11\},\{1,2,6,9,10,11\},\{1,2,3,7,10,11\},\{1,2,3,4,8,11\}]$$

Oh my god, I just realized I have $V6$ wrong, because $\{1,2,3,4,8,11\}$ was deemed a numerical semigroup and it isn't. I addressed this above now too! There's also a clear pattern for the semigroups we've tested, but it's possible this fails at some point in bigger sets.

Conjecture 8 *The Cycle of a numerical semigroup contains at most two numerical semigroups.*

But really, what all of this shows is something somewhat clear from the beginning: Numerically semigroups are algebraic structures. Doing swaps of 1s that pay no respect to the algebraic structure are unlikely to produce a bijection/injection.

7 Saturated Subsets

A subset of a ring is called saturated if it's closed under taking divisors. Since numerical semigroups are additively closed, their complements are saturated. As my point of view is often the set of gaps (i.e. when I refer to a numerical semigroup, I'm often referring to its set of gaps), this means numerical semigroups are a subset of saturated subsets of a certain size. Let's list some of these.

$$\{1\}$$

$$\{1,2\},\{1,3\}$$

$$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,5\}$$

$$\{1,2,3,4\},\{1,2,3,5\},\{1,2,3,6\},\{1,2,3,7\}$$

$$\{1,2,4,5\},\{1,2,4,7\},\{1,2,5,7\},\{1,3,5,7\}$$

$$\{1,2,3,4,5\},\{1,2,3,4,6\},\{1,2,3,4,7\},\{1,2,3,4,8\}$$

$$\{1,2,3,4,9\},\{1,2,3,5,6\},\{1,2,3,5,7\},\{1,2,3,5,9\}$$

$$\{1,2,4,5,7\},\{1,2,4,5,8\},\{1,2,4,7,8\},\{1,3,5,7,9\}$$

As you can see, numerical semigroups take up a lot of this space! Let's try to figure out how many are as the size gets larger. First of all, we get a clear size restriction. The maximal gaps under division are called the *fundamental gaps*, for example, in this paper.

Letting $\{f_1,\ldots,f_t\}$ be the set of fundamental gaps of S, we then have

$$S^c = \bigcup_{i=1}^{t} Div(f_i)$$

where $Div(f_i)$ is the set of divisors of f_i. By inclusion-exclusion, this means

$$g = \sum_{\emptyset \neq J \subset \{1,2,\ldots,t\}} (-1)^{|J|+1} \left| \bigcap_{j\in J} Div(f_j) \right|$$

The intersection of sets of divisors is exactly the set of divisors of the *gcd*, and as each f_i are fundamental divisors, the *gcd* of any two of them will be 1 - THIS IS FALSE. Take $S=\{1,2,3,4,5,7,9,10,11,13\}$. Then the fundamental gaps are $\{13, 11, 10, 9, 7, 4\}$. So the following only holds when the fundamental gaps are pair-wise relatively prime. Hence,

$$g = \sum_{i=1}^{t} |Div(f_i)| + \sum_{\substack{J \subset \{1,2,\dots,t\} \\ |J| \geq 2}} (-1)^{|J|+1}$$

$$\sum_{i=1}^{t} \sigma_0(f_i) + \sum_{k=2}^{t} \binom{t}{k} (-1)^{k+1}$$

where, as we started in part 1, we have $\sigma_k(n) = \sum_{d|n} d^k$, so $\sigma_0(n)$ is the number of divisors of n.

The alternating sum of binomial coefficients is a classic one: First recall

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

So setting $a = -1$ and $b = 1$, we get that

$$0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

Then

$$\sum_{k=2}^{n} \binom{n}{k} (-1)^{k+1} = -\binom{n}{1} + \binom{n}{0} = 1 - n$$

Then we get

$$g = 1 - t + \sum_{i=1}^{t} \sigma_0(f_i)$$

For example, consider

$$\{1, 2, 3, 5, 6, 7\}$$

Then the fundamental gaps are $\{5, 6, 7\}$, and

$$6 = 1 - 3 + 2 + 4 + 2$$

And the great thing is that any set of $t \leq g-1$ relatively prime integers between 1 and $2g - 1$ that satisfy the equation will give a saturated subset.

CORRECTION: Doing this sum with $S = \{1, 2, 3, 4, 5, 7, 9, 10, 11, 13\}$ gives

$$1 - 6 + (2 + 2 + 4 + 3 + 2 + 3) = 11,$$

while $g = 10$. This is because the above sum only works when the fundamental gaps are pair-wise relatively prime.

To get a feel for how many fundamental gaps numerical semigroups tend to have, let's look at the statistic generating function:

$$g = 1 \ : \ x$$

$$g = 2 \ : \ 2x$$

$$g = 3 \ : \ 3x^2 + x$$

$$g = 4 \ : \ 3x^3 + 3x^2 + x$$

$$g = 5 \ : \ x^4 + 6x^3 + 5x^2$$

To isolate the t part, let's write the equation as

$$g - 1 = \sum_{i=1}^{t} (\sigma_0(f_i) - 1)$$

If we look at the product

$$\prod_p \left(1 + x^{\sigma_0(p)-1} + x^{\sigma_0(p^2)-1} + \dots\right)$$

then the coefficient of N is the number of prime power tuples $(p_1^{k_1}, p_2^{k_2}, \dots, p_t^{k_t})$ for which

$$N = \sum_{i=1}^{t} (\sigma_0(p_i^{k_i}) - 1)$$

But this isn't exactly what we want, since a set of pair-wise relatively prime numbers does not have to contain only prime powers. But if the equality above holds, then we know that $(p_1^{k_1}, \dots, p_t^{k_t})$ could come from any choice of grouping of these primes.

Laszlo Toth has a nice paper on the probability that k integers are relatively prime:

$$\prod_p \left(1 - \frac{1}{p}\right)^{k-1} \left(1 - \frac{k-1}{p}\right)$$

by analyzing the sum

$$\sum_{\substack{1 \le a_1 < \dots < a_k \le n \\ gcd(a_i, a_j) = 1}} 1$$

If we instead look at

$$\sum_{\substack{1 \le a_1 < \dots < a_k \le 2g-1 \\ gcd(a_i, a_j) = 1}} \prod_{i=1}^{k} x^{\sigma_0(a_i)-1}$$

then the coefficient of x^{g-1} should be exactly the number of saturated subsets of $\{1, 2, \dots, 2g-1\}$ of size g. Toth includes another variable u, for which various values of u correspond to classical number theoretic functions. So let's define

$$P_k^{(u)}(n) = \sum_{\substack{1 \le a_1 < \dots < a_k \le n \\ gcd(a_i, a_j) = 1 \\ gcd(a_i, u) = 1}} \prod_{i=1}^{k} x^{\sigma_0(a_i)-1}$$

The first thing he does is prove a recurrence - let's see how that goes here:

$$P_{k+1}^{(u)}(n) = \sum_{\substack{a_{k+1}=1 \\ gcd(a_{k+1},u)=1}}^{n} \sum_{\substack{1\le a_1<\cdots<a_k\le n \\ gcd(a_i,a_j)=1 \\ gcd(a_i,u)=1 \\ gcd(a_i,a_{k+1})=1}} x^{\sigma_0(a_{k+1})-1} \prod_{i=1}^{k} x^{\sigma_0(a_i)-1}$$

$$\sum_{\substack{a_{k+1}=1 \\ gcd(a_{k+1},u)=1}}^{n} x^{\sigma_0(a_{k+1})-1} \sum_{\substack{1\le a_1<\cdots<a_k\le n \\ gcd(a_i,a_j)=1 \\ gcd(a_i,u)=1 \\ gcd(a_i,a_{k+1})=1}} \prod_{i=1}^{k} x^{\sigma_0(a_i)-1} = \sum_{\substack{a_{k+1}=1 \\ gcd(a_{k+1},u)=1}}^{n} x^{\sigma_0(a_{k+1})-1} P_k^{(ua_{k+1})}(n)$$

Which gives the analogue of his lemma 1:

Lemma 1 *For all $k, n, u \geq 1$, we have*

$$P_{k+1}^{(u)}(n) = \sum_{\substack{j=1 \\ gcd(j,u)=1}}^{n} x^{\sigma_0(j)-1} P_k^{(uj)}(n)$$

Unfortunately, the rest of the proof is induction, which means there is little insight into where the coefficients came from. His main theorem is that for a fixed k, we have uniformally for $n, u \geq 1$,

$$P_K^{(u)}(n) = A_k f_k(u) n^k + O(\theta(u) n^{k-1} log^{k-1} n),$$

with

$$A_k = \prod_p \left(1 - \frac{1}{p}\right)^{k-1} \left(1 - \frac{k-1}{p}\right)$$

and

$$f_k(u) = \prod_{p|u} \left(1 - \frac{k}{p+k-1}\right)$$

and $\theta(u)$ is the number of square-free divisors of u.

What should the analogues of these values be? Another thing is that we don't actually care for fixing k and letting n, u get bigger. We want to fix $n = 2g - 1$ and let k be free. Then we're specifically interested in the coefficient of x^{g-1}.

Maybe the recurrence can help us. On the right, the coefficient of x^{g-1} will come from a term in $P_k^{(uj)}(2g-1)$ with exponent $g - \sigma_0(j)$. Which is saying we get a tuple adding to $g - 1$ of length $k + 1$ if we have a k-tuple and some j relatively prime to u and the tuple's elements, so that the sum of the tuple is $g - \sigma_0(j)$.

8 Another Point of View

I mentioned back in part 1 that numerical semigroups can be thought of in terms of certain partitions. I want to go through that more deeply now. The partition associated to a numerical semigroup is usually flipped so the largest part is on top, but I want to do it this way for reasons that will become apparent soon. Here's an example for $A = \langle 3, 4 \rangle$ with hooks $H_A = \{1^2, 2^2, 5\}$.

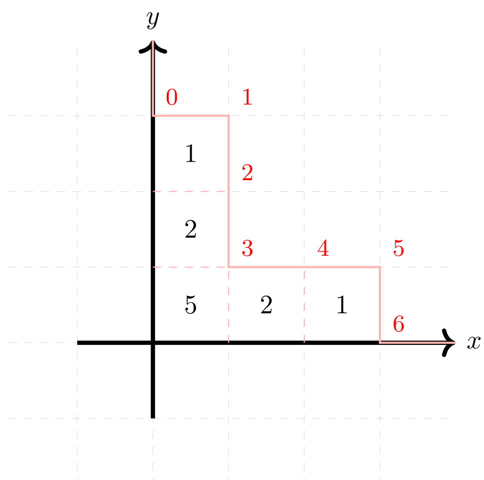


Figure 7: The partition associated with $A = \langle 3, 4 \rangle$, whose complement is $\{1, 2, 5\}$.

There are tons of questions you can ask about this correspondence. Why does the hookset correspond to the complement of A? What do the multiplicities tell us? Which partitions come from numerical semigroups? How many? For a positive integer m, a partition is called a m-core if none of its hooklengths are divisible by m. The p-cores for prime p correspond is some way to irreducible representations of the symmetric group, but I haven't read much about that yet.

What's nice is that any partition coming from a numerical semigroup A with minimal generating set $\mathcal{A} = \{a_1, \ldots, a_k\}$ is a simultaneous $\mathcal{A}$-core partition, since the hookset must be A^c and therefore cannot divide any of the a_i. Some work (summarized nicely in poster form with paper on arXiv) has been done on this by Hannah Constantin and Benjamin Houston-Edwards at Yale with Nathan Kaplan. This includes giving some estimates on how many (a, b)-core partitions come from numerical semigroups (they expect 0).

I want to think about the construction of this partition as an embedding $\phi_A : \mathbb{R} \to \mathbb{R}^2$ governed by the numerical semigroup A. We define ϕ_A by its derivative, so that the function is defined up to a shift and we can "standardize" the partition of A so its bottom left corner is the unit cube with corners (0,0) and (1,1). So

$$\phi'_A(x) = \begin{cases} -\infty & \lfloor x \rfloor \notin A \\ 0 & \lfloor x \rfloor \in A \end{cases}$$

Notice that we move downwards for every element not in A, so the number of rows corresponds to the genus of A. Since $A \subset \mathbb{N}$, we can choose $\phi_A(0) = g(A)$ to get a “standardized" partition. This results in $\phi_A(x) = 0$ for all $x \in [F(A) + 1, \infty)$.

But since we're only sitting in the positive quadrant, we've gotten an embedding $\phi_A : \mathbb{N} \to \mathbb{N}^2$ that splits the codomain into two parts. So while we've just been looking at the partition a numerical semigroup gives us (the inside), we could also look at the *adeal* the numerical semigroup gives us (the outside)! Let $\pi(A)$ denote the first and $\alpha(A)$ denote the latter.

And I'll define the notion of an adeal, which is the additive analogue of an ideal that Trevor Hyde and I worked on. A subset A of a semiring S is called an *adeal* if $S + A \subset A$. The set S is the *trivial adeal.* We'll denote the set of adeals of S as $ad(S)$. A quick proposition about adeals to get a feel for them:

Proposition 4 *Let S be a semiring and A an adeal. Then*

- *A is trivial if and only if $0 \in A$.*
- *S is a ring if and only if $ad(S) = \{S\}$.*

Anyway, to simplify notation, let's think of $\mathbb{N}^2$ as $\{x^n y^m \mid n, m \in \mathbb{N}\}$, which turns adeals of $\mathbb{N}$ into multiplicatively closed sets in $\mathbb{N}[x]$.

$$A = \langle 1 \rangle, \quad \pi(A) = \emptyset, \quad \alpha(A) = \langle 1 \rangle$$
$$A = \langle 2, 3 \rangle, \quad \pi(A) = 1, \quad \alpha(A) = \langle x, y \rangle$$
$$A = \langle 2, 2g{+}1 \rangle, \quad \pi(A) = g{+}(g{-}1){+}\cdots{+}2{+}1, \quad \alpha(A) = \langle y^g, xy^{g-1}, \ldots, x^{g-1}y, x^g \rangle$$

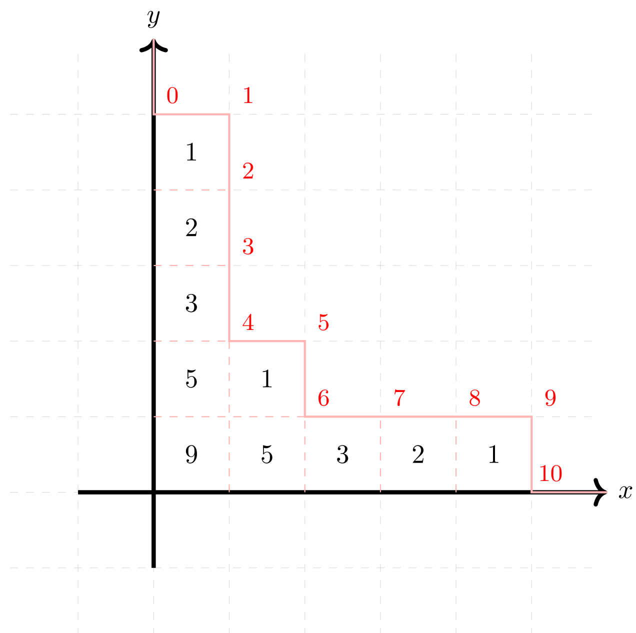


Figure 8: Given $A = \langle 4, 6, 7 \rangle \in N$, we have $\pi(A) = 5 + 2 + 1 + 1 + 1$ and $\alpha(A) = \langle y^5, xy^2, x^2y, x^5 \rangle$

This paper by Keith and Nath seems to be a really good place to look at this connection. They mention the following theorems.

Theorem 7 *Let $S = \{s_1, \ldots, s_k\}$ be any set of positive integers. The set of partitions which are simultaneous s_i-core for all $s_i \in S$ is finite if and only if* $\gcd(S) = 1$.

Theorem 8 *Among all partitions, the hooksets of length g are exactly the complements of numerical semigroups of genus g.*

This means we can think of a numerical semigroup $S = \langle a_1, \ldots, a_{e(S)} \rangle$ as a smooth curve in $\mathbb{N}^2$, cutting it into two connected components. One component is the $\{a_1, \ldots, a_{e(S)}\}$-core partition $\pi(S)$ and the other component is the adeal $\alpha(S)$ whose generators are indicated by those elements $n \in S$ such that $n-1 \notin S$. This means we can view S *almost* as a literal partition of the plane, though they intersect in the path that originally defined them of course.

$$\mathbb{N} \times \mathbb{N} = \pi(S) \cup \alpha(S)$$

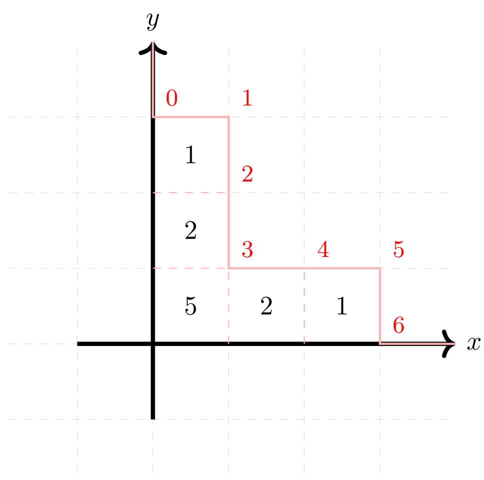
$$\pi(S) \cap \alpha(S) = \phi_S(S).$$

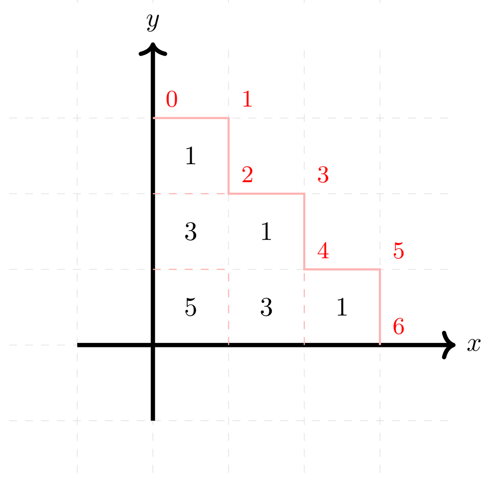
For example, we know that $y^{g(S)}$ is the first minimal generator of $\alpha(A)$ that we pick up as we walk along our path in the negative integers towards 0. And

$x^{F(S)-g(S)+1}$ will always be the last minimal generator, since by definition, we contain all elements larger than $F(S)$.

Supposing S to be nontrivial, the second time we pick up a minimal generator is the first time $[1, m] \not\subset S^c$, which is exactly the multiplicity of S! You can check this by looking at the two figures, noting that the number in the corner is in fact the multiplicity. So the second minimal generator we have is $xy^{g(S)+1-m(S)}$.

While people have somewhat studied $\pi(S)$, I've never seen someone look at the complement and study $\alpha(S)$, even though its minimal generating set tells you the genus, Frobenius number, and multiplicity of S. Note that these three things don't determine a numerical semigroup - take $\langle 3, 4\rangle^c = \{1, 2, 5\}$ and $\langle 2, 7\rangle^c = \{1, 3, 5\}$, for example. Their respective partitions and adeals are





$$\alpha(\langle 3, 4\rangle) = \langle y^3, xy, x^3\rangle,$$
$$\alpha(\langle 2, 7\rangle) = \langle y^3, xy^2, x^2y, x^3\rangle.$$

In general, since the staircase of leg length g corresponds to $\langle 2, 2g+1\rangle$, we have

$$\alpha(\langle 2, 2g+1\rangle) = \langle y^g, xy^{g-1}, \ldots, x^{g-1}y, x^g\rangle,$$

which is the basis for homogeneous degree g polynomials in two variables!

If we call the number of minimal generators of an adeal its dimension, then it is natural to ask what the dimension of $\alpha(S)$ tells us about S. From the examples we've seen,

$$\dim(\alpha(\mathbb{N})) = 1$$
$$\dim(\alpha(\langle 2, 3\rangle)) = 2$$
$$\dim(\alpha(\langle 2, 2g+1\rangle)) = g+1$$
$$\dim(\alpha(\langle 3, 4\rangle)) = 3$$
$$\dim(\alpha(\langle 4, 6, 7\rangle)) = 4$$

We will pick up a minimal generator for each $n \notin S$ with $n+1 \in S$. Such an element would lie in $S\triangle(1+S)$, the symmetric difference of these two sets. Which I love, since we've been working with symmetric differences a lot recently! However, we would also get more:

$$\mathbb{N}\triangle(1+\mathbb{N}) = \{0, 1\}$$
$$\langle 2, 3\rangle\triangle(1+\langle 2, 3\rangle) = \{0, 1, 2\}$$
$$\langle 2, 2g+1\rangle\triangle(1+\langle 2, 2g+1\rangle) = \{0, 1, 2, \ldots, 2g\}$$
$$\langle 3, 4\rangle\triangle(1+\langle 3, 4\rangle) = \{0, 1, 3, 5, 6\}$$
$$\langle 4, 6, 7\rangle\triangle(1+\langle 4, 6, 7\rangle) = \{0, 1, 4, 5, 6, 9, 10\}$$

Since $S\triangle(1+S)$ encodes when containment in S changes as we move along the integers, it exactly encodes the corners of our partition, which determines the whole partition. Define the generating function for this symmetric difference to be

$$D_S(x) = \frac{1}{2} + \frac{1}{2}\sum_{n\in S\triangle(1+S)} x^n.$$

Note that this polynomial could be defined in terms of the Hilbert series of S, as defined in this paper by Alfonsin and Rodseth, and in fact can be generalized for any shift $S\triangle(m+S)$. For $m = m(S)$, this essentially reduces to the polynomial they get when writing the Hilbert series is terms of its Apery set.

Proposition 5 *Let $S \subseteq \mathbb{N}$ be a proper numerical semigroup. Then*

$$\dim(\alpha(S)) = D_S(1)$$

Proof 3 *The elements in the symmetric difference begin with $0 \in S$ and end with $F(S)+1 \in S$. If we order $S\triangle(1+S)$ to be in increasing order, then this means that the elements of the symmetric difference alternate in containment in S. And since they start and end contained in S, they must switch an even number of times, and the equation follows.*

A small remark that's obvious but interesting is that $\deg(D_S(x)) = F(S) + 1$. Let's see a few examples.

$$D_{\langle 2,3\rangle}(x) = \frac{2 + x + x^2}{2}$$

$$D_{\langle 2,2g+1\rangle}(x) = \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{2g} x^n = \frac{1}{2} + \frac{1}{2}\left(\frac{x^{2g+1} - 1}{x - 1}\right) = \frac{x^{2g+1} + x - 2}{2(x-1)}$$

$$D_{\langle 2,2g+1\rangle}(1) = g + 1$$

$$D_{\langle 2,2g+1\rangle}(-1) = 1$$

Let $S = \langle 3, 4\rangle$, then

$$D_S(x) = \frac{2 + x + x^3 + x^5 + x^6}{2}.$$

$$D_S(1) = 3$$

$$D_S(-1) = 0$$

Let $S = \langle 4, 6, 7\rangle$, then

$$D_S(x) = \frac{2 + x + x^4 + x^5 + x^6 + x^9 + x^{10}}{2}.$$

$$D_S(1) = 4$$

$$D_S(-1) = 1$$

Let's look at a non-symmetric example of $S = \langle 3, 7, 8\rangle$.

$$D_S(x) = \frac{2 + x + x^3 + x^4 + x^6}{2}.$$

$$D_S(1) = 3$$

$$D_S(-1) = 1$$

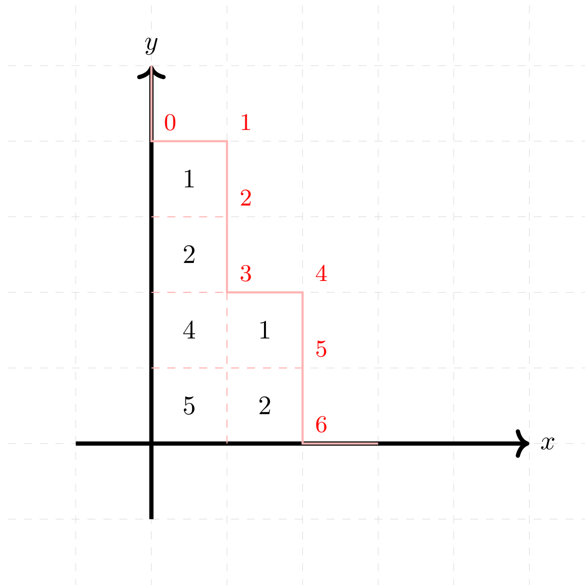


Figure 9: With $S = \langle 3, 7, 8 \rangle \subset \mathbb{N}$, we have $\pi(S) = 2 + 2 + 1 + 1$ and $\alpha(S) = \langle y^4, xy^2, x^2 \rangle$

Can we think of an interpretation for $D_S(-1)$ in terms of the numerical semigroups/partitions/adeals? Well, first, by definition,

$$D_S(-1) = \frac{1}{2} + \frac{1}{2} \sum_{n \in S \triangle (1+S)} (-1)^n$$

So a direct interpretation would be that this is

$$\frac{1}{2}(1 + EC(S) - OC(S))$$

where EC is the number of even corners of S, and OC is the number of odd corners of S.

We've seen already that the numerical semigroup $\{1, 3, 5, \ldots, 2g-1\}^c$ will be a staircase:

$$\pi = g + (g-1) + \cdots + 1$$
$$\alpha = \langle y^g, xy^{g-1}, \ldots, x^{g-1}y, x^g \rangle$$

How about the ordinary semigroup $\mathcal{O}_g = \{1, 2, \ldots, g\}^c$? It's a single column:

$$\pi(\mathcal{O}_g) = g$$

$$\alpha(\mathcal{O}_g) = \langle y^g, x \rangle$$

9 The Partition Function of a Semigroup

In this section, I want to look at how the shifting operation $\phi : \mathcal{N}_g \to \mathcal{N}_{g+1}$ interacts with the partition point of view and look at partition functions $D_S(x)$ under shifts. I also want to look at the characterization of numerical semigroup by adeal dimension.

The nice thing is that shifting a set keeps its shape (its difference set), so it should interact nicely with the partitions. First, the dimension of $\alpha(S)$ is the number of inner corners in its (Young tableau?) partition. $D_S(1)$ counts the total number of corners, which must alternate "inner, outer, inner, …, outer, inner", hence the $+1$ and division by 2 gives the number of inner corners. $D_S(-1)$ will do the same, except add 1 if that corner is even and -1 if it's odd. So it will be the number of even corners minus the number of odd corners.

Looking at the wiki page for acyclic orientations (which is what $|\chi_G(-1)|$ counts for a graph G), they actually also talk about partial cubes (partial hamming graphs)! In particular, they note that the set of acyclic orientations form a partial cube because they can be connected when their orientations differ in one edge.

Anyway, given the fact that

$$D_S(-1) = \frac{1}{2}(1 + EC(S) - OC(S))$$

an immediate thing to do would be to show that this is even an integer. Meaning, is it true that $EC(S) - OC(S)$ must always be odd? We start at 0, go to 1, and then drop a certain amount. If the straight-edge-lengths between points is odd, it switches parity, and if it's even, it stays the same parity. Let's look at one more example with $g = 5$ and $F(S) = 6$:

$$S = \{1, 2, 3, 5, 6\}^c$$

$$S \triangle (1 + S) = \{0, 4, 7, 8, 9, 10, \dots\} \triangle \{1, 5, 8, 9, \dots\} = \{0, 1, 4, 5, 7\}$$

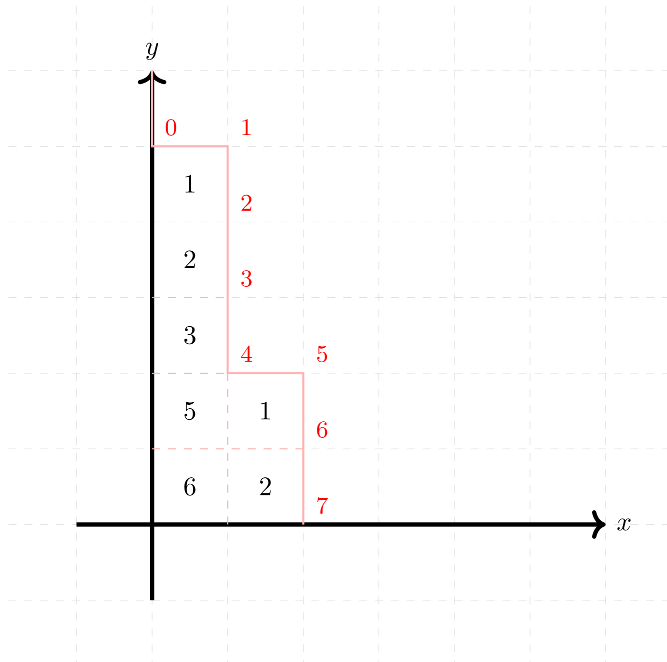


Figure 10: $\pi(S) = 5 + 2 \quad \alpha(S) = \langle y^5, xy^5, xy^2, x^2y^2, x^2 \rangle$

Note the adeal we immediately get is $\alpha(S) = \langle y^5, xy^5, xy^2, x^2y^2, x^2 \rangle$, but we can remove $x^2y^2 = x(xy^2)$ and $xy^5 = x(y^5)$, so a minimal generating set will be

$$\alpha(S) = \langle y^5, xy^2, x^2 \rangle$$

which correspond, again, to the inner corners. In the actual numerical semigroup, an inner corner s is a term so that $s \in S$ and $s - 1 \notin S$. This set of inner corners completely determines the numerical semigroup, so we get a way to think of semigroups as adeals in $\mathbb{N}^2$. But before doing that, it's very difficult (in my opinion) to deduce the adeal generators, so let's see if we can form a useful explicit bijection

$$\{s \in S \mid s - 1 \notin S\} \longleftrightarrow \{(a, b) \mid x^a y^b \textit{ minimal generator of } \alpha(S)\}$$

Clearly $0 \leftrightarrow y^g$ and $F(S) + 1 \leftrightarrow y^{D_S(1)}$. So what happens in between? Well each $s' \in \{s \in S \mid s - 1 \notin S\}$ is the start of a new section, separated from the past by at least one gap. Looking at the partition, the inner corners (except 0) are all below an outer corner.

And in reality, it's easier to form a bijection between all corners and $S \triangle (1 + S)$ and then take every other term. Let $S \triangle (1 + S) = \{s'_0, s'_1, \ldots, s'_t\}$. Then $0 = s'_0 \leftrightarrow (0, g)$ and $1 = s'_1 = (1, g)$. In general, the length of the edge connecting two corners is $s'_i - s'_{i-1}$. So the next corner is

$$(s'_1 - s'_0, g - (s'_2 - s'_1)),$$

then

$$(s_1' - s_0', g - (s_2' - s_1'))$$

$$((s_1' - s_0') + (s_3' - s_2'), g - (s_2' - s_1'))$$

$$((s_1' - s_0') + (s_3' - s_2'), g - (s_2' - s_1') - (s_4' - s_3'))$$

Looking at this last one expanded:

$$(-s_0' + s_1' - s_2' + s_3',\ g + s_1' - s_2' + s_3' - s_4')$$

A clear pattern emerges! Let's define

$$p_k = \sum_{i=1}^{k} (-1)^{i-1} s_i'$$

Then our points go

$$(0, g)$$

$$(p_1, g)$$

$$(p_1, g + p_2)$$

$$(p_3, g + p_2)$$

$$(p_3, g + p_4)$$

$$(p_5, g + p_4)$$

and so on. This means that the general form of an inner corner is $(p_{2k-1}, g+p_{2k})$. It's important to recognize that p_n is positive for odd n and negative for even n. So the process goes:

1. Compute $S \triangle (1 + S) = \{s_0', s_1', \ldots, s_t'\}$
2. Compute $p_1, p_2, \ldots, p_{t-1}$, and consider $p_0 = 0$.
3. Then $\alpha(S) = \langle x^{p_k} y^{g+p_{k+1}}\ :\ 1 \leq k \leq t-2,\ odd\rangle$ along with y^g.

Let's try this:

1. $S = \{1\}$
2. $S \triangle (1 + S) = \{0, 1, 2\}$
3. $p_0 = 0, p_1 = 1, p_2 = -1$
4. Then $\alpha(S) = \langle y, x\rangle$

---

1. $S = \{1, 2\}$
2. $S \triangle (1 + S) = \{0, 1, 3\}$

3. $p_0 = 0, p_1 = 1, p_2 = -2$
4. Then $\alpha(S) = \langle y^2, x \rangle$

---

1. $S = \{1, 3\}$
2. $S \triangle (1 + S) = \{0, 1, 2, 3, 4\}$
3. $p_0 = 0, p_1 = 1, p_2 = -1, p_3 = 2, p_4 = -2$
4. Then $\alpha(S) = \langle y^2, xy, x^2 \rangle$

Nice, this was very helpful! Let's continue.

1. $S = \{1, 2, 3\}$
2. $S \triangle (1 + S) = \{0, 1, 4\}$
3. $p_0 = 0, p_1 = 1, p_2 = -3$
4. Then $\alpha(S) = \langle y^3, x \rangle$

---

1. $S = \{1, 2, 4\}$
2. $S \triangle (1 + S) = \{0, 1, 3, 4, 5\}$
3. $p_0 = 0, p_1 = 1, p_2 = -2, p_3 = 2, p_4 = -3$
4. Then $\alpha(S) = \langle y^3, xy, x^2 \rangle$

---

1. $S = \{1, 2, 5\}$
2. $S \triangle (1 + S) = \{0, 1, 3, 5, 6\}$
3. $p_0 = 0, p_1 = 1, p_2 = -2, p_3 = 3, p_4 = -3$
4. Then $\alpha(S) = \langle y^3, xy, x^3 \rangle$

---

1. $S = \{1, 3, 5\}$
2. $S \triangle (1 + S) = \{0, 1, 2, 3, 4, 5, 6\}$
3. $p_0 = 0, p_1, p_2 = \pm 1, p_3, p_4 = \pm 2, p_5, p_6 = \pm 3$
4. Then $\alpha(S) = \langle y^3, xy^2, x^2y, x^3 \rangle$

An alternating sum always reminds me of an Euler characteristic, though I have to admit I know I've forgotten a lot of the cool connections the Euler characteristic gives us - so this will be a chance to refresh that memory. If I remember correctly, Sagan's interpretation of coefficients of the chromatic polynomial of a graph has something to do with constructing CW complexes and looking at their (co)homology groups. Let's look back at that.

He begins by defining a broken circuit for a graph G. First, we assign an ordering of the edges $E = \{e_1, \ldots, e_m\}$. A *broken circuit* of G is a subset $B \subseteq E$ of edges formed by removing the smallest (with respect to the ordering) from a circuit C. A subset $A \subseteq E$ is called a *no-broken-circuit set (NBC set)* if it contains no broken circuits as a subset. We define

$$nbc_k(G) = |\{A \subseteq E \mid A \textit{ is an NBC set with } k \textit{ edges}\}|$$

Sagan shows quite amazingly that these are exactly the coefficients of the chromatic polynomial!

$$P(G;t) = \sum_{k=0}^{n} (-1)^k nbc_k(G) t^{n-k}$$

His next step is to connect the similarly amazing identity

$$P(G;-1) = |\{\textit{acyclic orientations of } G\}|$$

to NBC sets. Sagan and Andreas Blass do this in another paper.

He then moves on to hyperplane arrangements. A hyperplane H is a subspace of $\mathbb{R}^n$ of dimension $n-1$. Often (i.e. knot theory), it's more helpful to think of this as *codimension 1*. Anyway, a hyperplane arrangement $\mathcal{H} = \{H_1, H_2, \ldots, H_t\}$ is just a finite set of hyperplanes. These all go through the origin, and hence their intersection will cut $\mathbb{R}^n$ into a number of pieces. The regions of $\mathcal{H}$ are these connected components after we remove all hyperplanes in $\mathcal{H}$ from $\mathbb{R}^n$, and we denote this by $R(\mathcal{H})$.

Please refer to Sagan's paper for many good examples and visuals.

The point is that since each hyperplane is codimension 1, it is generated by 1 equation $x_i = x_j$ (this is why codimension is nicer than dimension sometimes). This is great! It means we have a very natural way to think of a graph as a hyperplane arrangement. For a graph G, define

$$\mathcal{H}(G) = \{x_i = x_j \ : \ (i,j) \in E\}$$

Then it turns out that

$$P(G;-1) = (-1)^n |R(\mathcal{H}(G))|$$

which means each region corresponds to some acyclic orientation. To get an explicit bijection, we note that each hyperplane $x_i = x_j$ cuts $\mathbb{R}^n$ into two half-planes $x_i < x_j$ and $x_i > x_j$. We can think of this as an arc $i \to j$ or $j \to i$, i.e. an orientation of the edge (i,j).

Then the path is to show that each region in $R(\mathcal{H}(G))$ corresponds to an acyclic orientation of G. The first step to this is the *intersection lattice* of $\mathcal{H}$, which is the poset ordered by inclusion of intersections of some subset of $\mathcal{H}$. This includes the empty subset (which gives $\mathbb{R}^n$) and the whole subset (which gives the origin point). So this poset has a unique minimum $\hat{0}$, which means we can recursively define its Mobius function μ so that $\mu(\hat{0}) = 1$ and the sum of the Mobius function on any interval is always zero.

He makes the note, so I'll make it here: The usual "mobius function" in number theory is simply the Mobius function of the poset of positive integers ordered by divisibility. For any poset with a Mobius function, we can form a characteristic polynomial, and for the intersection lattice of a hyperplane arrangement, we get

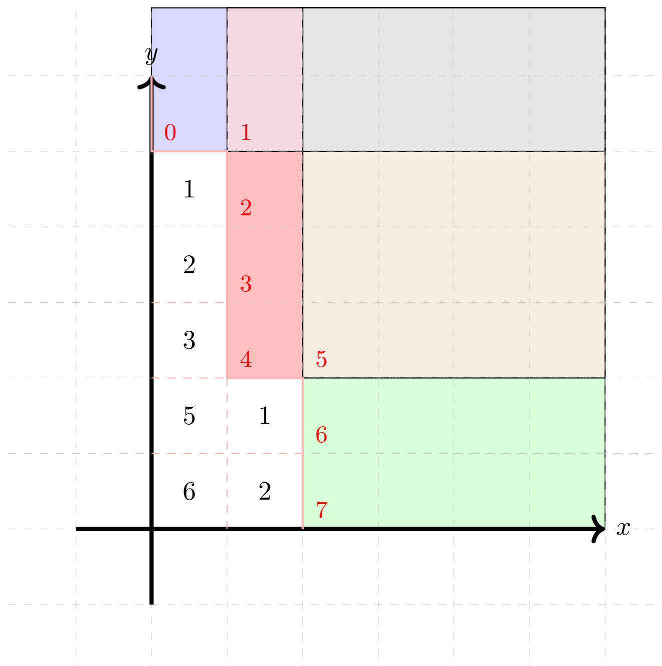
$$\chi(\mathcal{H};t) = \sum_{S \in L(\mathcal{H})} \mu(S) t^{dim(S)}$$

It turns out that for any hyperplane arrangement, $\chi(\mathcal{H};-1) = (-1)^n |R(\mathcal{H})|$, but if $\mathcal{H} = \mathcal{H}(G)$ for some graph G, then we also have

$$\chi(\mathcal{H}(G);t) = P(G;t)$$

It's slightly different than I remembered, but it's definitely what I was thinking of - the formation of hyperplane arrangements and dimensions in the intersection lattice to reform the chromatic polynomial is great! And it's a great direction to go for these!

Each inner corner gives us a square in the grid, and the adeal generated by each inner corner (i.e. the complement of the partition) is the union of all of these square. Let's visualize this:



If we instead take the intersection lattice of the adeal $\alpha(S)$, what do we get? Let's call it $L(S)$. It's helpful to note that $\langle x^a y^b \rangle < \langle x^{a'} y^{b'} \rangle$ in the poset if and only if $a \leq a'$ and $b \leq b'$. Also, the generator of the intersection of all adeals is that left-most point of the grey box. But we know our height is g and the length is $D_S(1) - 1$, so we get

$$\hat{0} = \langle x^{D_S(1)-1} y^g \rangle$$

For the above example,

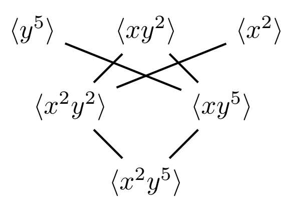


Figure 11: $L(\{1, 2, 3, 5, 6\})$

The characteristic polynomial of this poset is

$$\chi_S(t) = 1 + (-1)t + (-1)t + (0)t^2 + (1)t^2 + (0)t^2 = 1 - 2t + t^2 = (t-1)^2$$

And $\chi_S(-1) = 4$. For reference, let's also list

$$D_S(t) = \frac{1}{2} + \frac{1}{2} \sum_{n \in S \triangle (1+S)} t^n = \frac{2 + x + x^4 + x^5 + x^7}{2}$$

$$D_S(1) = 3$$

$$D_S(-1) = 0$$

Now let's see what changes with $S = \{1, 2, 3, 5, 7\}$. Then $S \triangle (1+S) = \{0, 1, 4, 5, 6, 7\}$, so $p_0 = 0, p_1 = 1, p_2 = -3, p_3 = 2, p_4 = -4, p_5 = 3$, and

$$\alpha(S) = \langle y^5, xy^2, x^2y, x^3 \rangle$$

The partition is

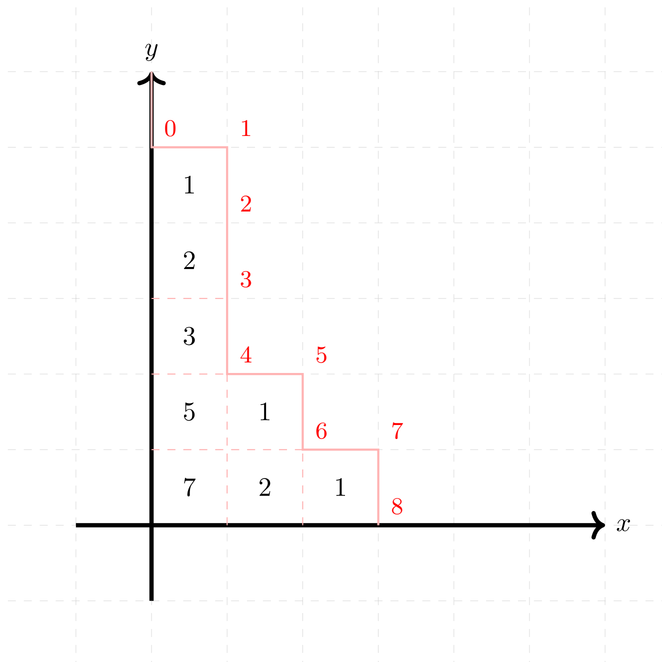


Figure 12: $S = \{1, 2, 3, 5, 7\}^c$

And $L(\{1, 2, 3, 5, 7\})$ isn't a ranked poset! But we can still define $dim(A) = d(A, \hat{0})$ and the characteristic polynomial. Before continuing, let's go ahead and enumerate $\mathcal{N}$ by α-dimension.

There is 1 numerical semigroups of α-dim 1, which is $\mathbb{N}$. For $dim_\alpha = 2$, we get an infinite number! For all $g \geq 1$, we have $dim_\alpha(\mathcal{O}_g) = 2$. And any other numerical semigroup will have more than 2 inner corners. For $\alpha = 3$, we get all

L numerical semigroups. For $f \geq g$, we can define the $L_{f,g}$ numerical semigroup as

$$L_{f,g} = \{1, 2, \ldots, g-1, f\}^c$$

Bras-Amoros calls these semigroups *almost-ordinary numerical semigroups* (in here, for example), and distinguishes them as numerical semigroups with a $m(S) = g$, which means there is a unique element after the gap, f.

So fixing an α-dimension doesn't restrict the genus. But fixing a genus g does indeed restrict the α-dimension. The numerical semigroup of genus g that maximizes inner corners will be the staircase semigroup $\{1, 3, \ldots, 2g-1\}$, which for genus g, will have $\alpha = g+1$. In general, let's define the *genus-alpha* number $n(g, \alpha)$ to be the number of numerical semigroups S with $dim_\alpha(S) = \alpha$ and $g(S) = g$.

We picture this as fixing a square in the grid, and counting the semigroups that achieve such bounds - height g and $\alpha - 2$ interior inner corners. Let's make a table! We know that for $\alpha > g+1$, we'll have $n(g, \alpha) = 0$. And we'll have $n(g, 2) = 1$ for all $g \geq 1$, from $\mathcal{O}_g$, and $n(g, g+1) = 1$ for all g, from the staircase semigroup. Finally, we naturally have

$$N(g) = \sum_{\alpha=1}^{g+1} n(g, \alpha)$$

$g \backslash \alpha$	1	2	3	4	5	6	7	8	9	10	11
0	1										
1	0	1									
2	0	1	1								
3	0	1	2	1							
4	0	1	4	1	1						
5	0	1	6	4	0	1					
6	0	1	9	8	4	0	1				

A notion that I think could be related is discussed in Gapsets and Numerical Semigroups by Eliahou and Fromentin, the *depth* of a numerical semigroup. This is defined as

$$depth(S) = \left\lceil \frac{F(S)+1}{m(S)} \right\rceil$$

so it's essentially a measure of how "spread apart" the intervals are between the initial sequence of gaps and the final sequence of non-gaps. For example,

$$depth(\mathcal{O}_g) = 1$$

On the other extreme,

$$depth(\{1, 3, \ldots, 2g-1\}) = \frac{2g}{2} = g$$

For the almost-ordinary numerical semigroups $L_{f,g}$, we have

$$depth(L_{f,g}) = \left\lceil \frac{f+1}{g} \right\rceil$$

This was also used by Mariam Dhayni in Wilf's Conjecture for Numerical Semigroups, though they did not explicitly name it as the depth. Also, Wilf's Conjecture is another thing we haven't talked about despite it being an incredibly foundational conjecture. But we'll get to that later!

It feels to me that a smaller depth would correspond to a smaller α-dimension. Recall $dim_\alpha(L_{f,g}) = 3$ for all f, g and $depth(L_{f,g})$ is maximized at $f = 2g - 1$, so

$$depth(L_{2g-1,g}) = \left\lceil \frac{2g}{g} \right\rceil = 2$$

and even if $f = g + 1$, the depth is 2, so the above holds for all f, g.

But we could have $depth(S) = 2$ without it being an L semigroup. For example, $\{1, 2, 3, 5, 6\}$ has $depth(S) = 2$ but $dim_\alpha(S) = 3$. Another good example to consider is the staircase semigroup $\{1, 3, \ldots, 2g - 1\}$, which has depth g and $dim_\alpha(S) = g + 1$.

Going to $\{1, 2, 3, 5, 7\}$, we get an example where $dim_\alpha(S) = 4$ and $depth = \lceil 8/4 \rceil = 2$, so the α-dimension exceeds the depth by 2. The depth depends on where we stop our first descent in the partition and on how long the path is. The alpha-dimension depends on the number of continuous intervals contained in S^c.

$$S^c = \{1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 13, 15\}$$

Go along with building up this numerical semigroup and see what happens to depth and α-dimension. Up to $\{1, 2, 3, 4, 5, 6\}$, we have $\alpha = 2$ and $depth = 1$. Adding on $\{8, 9\}$ changes $\alpha = 3$ and our depth becomes $\lceil 10/7 \rceil = 2$. Then we add $\{11, 12, 13\}$ and increase $\alpha = 4$, while $depth = 2$ remains. In other words, increasing the α-dimension forces a weak increase of depth. Adding 15 increases the depth to 3 but also increases α to 5.

And that's the key! The only thing that matters here is the first two gaps. What happens if we build up the staircase semigroup? $\{1\}$ starts with $\alpha = 2$ and $depth = 1$. Adding a 3 increases $\alpha = 3$ and $depth = 2$. Adding a 2 instead keeps $\alpha = 2$ but $depth = 1$. As such, we get an upper bound.

We can also get a lower bound with the following lemma.

Lemma 2 *For any numerical semigroup S, we have*

$$F(S) \geq m(S) - 1 + 2(dim_\alpha(S) - 2)$$

Proof 4 *We must have $dim_\alpha(S) - 1$ continuous intervals in S^c. Assuming we have the initial sequence of gaps up to $m(S) - 1$, the smallest way to get the remaining intervals is a single element. So starting at $m(S) - 1$, we'd add on every other number to get the $dim_\alpha(S) - 2$ remaining intervals. So the smallest possible Frobenius number is*

$$m(S) - 1 + 2(dim_\alpha(S) - 2)$$

And this bound is tight. For example, with the staircase partition,

$$2g - 1 \geq 1 + 2(g + 1 - 2) = 2g - 1$$

Also, we could expand this more by taking into account the genus, but for now, this will do!

Theorem 9 *For all non-trivial numerical semigroups S, we have*

$$1 + \frac{2(dim_\alpha(S) - 2)}{m(S)} \leq depth(S) \leq dim_\alpha(S) - 1$$

Proof 5 ***Lower Bound:*** *From the lemma, we have*

$$F(S) \geq m(S) - 1 + 2(dim_\alpha(S) - 2)$$

Adding one to both sides and dividing by $m(S)$ gives

$$\frac{F(S) + 1}{m(S)} \geq 1 + \frac{2(dim_\alpha(S) - 2)}{m(S)}$$

and so

$$depth(S) \geq \frac{F(S) + 1}{m(S)} \geq 1 + \frac{2(dim_\alpha(S) - 2)}{m(S)}$$

Upper Bound: *As we noted, for the initial non-trivial semigroup $\{1\}^c$, we have $\alpha = 2$ and depth $= 1$, so the inequality is satisfied. We can build any numerical semigroup up one by one from its gaps and see how the α and depth change as we do so. In essence, we're doing a mini induction for each semigroup separately.*

If we build the initial sequence of gaps, we know we have depth $= \alpha - 1$. So let's suppose we've built up a subset with at least two continuous intervals, $S_t^c := \{1, a_2, \ldots, a_t\} \subset S^c$ and $depth(S_t^c) \leq \dim_\alpha(S_t^c) - 1$.

Then what can happen when we add a_{t+1}? The Frobenius number increases to a_{t+1}, but the multiplicity stays the same, so the depth goes from $\lceil a_t/m(S) \rceil$ to $\lceil a_{t+1}/m(S) \rceil$.

Case 1: *If $a_{t+1} > a_t + 1$, then α increases by 1, while depth increases only if $a_t/m(S)$ is below some integer and $a_{t+1}/m(S)$ is above that integer.*

To show depth is weakly increasing (which means the inequality stays satisfied), we need to show that the depth can never increase by 2. If it did, then $a_t/m(S) = depth(S_t) - \gamma_0$ for some $\gamma_0 \in (0, 1)$, while $a_{t+1}/m(S) = depth(S_t) + 2 - \gamma_1$, for some $\gamma_1 \in (0, 1)$. So

$$a_{t+1} = depth(S_t)m(S) + 2m(S) - \gamma_1 m(S)$$

and

$$a_t = depth(S_t)m(S) - \gamma_0 m(S)$$

Subtracting these, we have

$$a_{t+1} - a_t = (2 + \gamma_0 - \gamma_1)m(S) \geq (1 + \varepsilon)m(S) > m(S)$$

So $a_{t+1} > a_t + m(S)$, which means there is some $s > a_t$ (which implies $s \in S$), for which $a_{t+1} = s + m(S)$. As both of these are elements of S, their sum must be too, contradicting the fact that S is a numerical semigroup.

Case 2: *If $a_{t+1} = a_t + 1$, then α does not increase, and the only way depth increases is if $(a_t + 1)/m(S) = depth(S_t) - \gamma_0$ and $(a_{t+1} + 1)/m(S) = depth(S_t) + \gamma_1$. Then*

$$a_t = depth(S_t)m(S) - \gamma_0 m(S) - 1$$

$$a_{t+1} = depth(S_t)m(S) + \gamma_1 m(S) - 1$$

Subtracting the top from the bottom gives

$$1 = (\gamma_1 + \gamma_0)m(S)$$

At first that seems like nothing, but analyzing $\gamma_0, \gamma_1 \in (0, 1)$ reveals an issue! Because the actual interval is shorter, as we're dividing by $m(S)$, we actually have $\gamma_0, \gamma_1 \in \left[\frac{1}{m(S)}, \frac{m(S)-1}{m(S)}\right]$, so

$$\gamma_0 + \gamma_1 \geq \frac{2}{m(S)}$$

and so

$$(\gamma_1 + \gamma_0)m(S) \geq 2 > 1,$$

which gives the contradiction.

In the following table, let's see how tight these bounds are. I'll highlight in blue the rows that have different depth and α.

S	Lower	$depth(S)$	Upper
$\{1\}$	1	1	1
$\{1,2\}$	1	1	1
$\{1,3\}$	2	2	2
$\{1,2,3\}$	1	1	1
$\{1,2,4\}$	$5/3$	2	2
$\{1,2,5\}$	$5/3$	2	2
$\{1,3,5\}$	3	3	3
$\{1,2,3,4\}$	1	1	1
$\{1,2,3,5\}$	$3/2$	2	2
$\{1,2,3,6\}$	$3/2$	2	2
$\{1,2,3,7\}$	$3/2$	2	2
$\{1,2,4,5\}$	$5/3$	2	2
$\{1,2,4,7\}$	$7/3$	3	3
$\{1,3,5,7\}$	4	4	4
$\{1,2,3,4,5\}$	1	1	1
$\{1,2,3,4,6\}$	$7/5$	2	2
$\vdots$		$\vdots$	$\vdots$
$\{1,2,3,4,9\}$	$7/5$	2	2
$\{1,2,3,5,6\}$	$3/2$	2	2
$\{1,2,3,5,7\}$	2	2	3
$\{1,2,3,5,9\}$	2	3	3
$\{1,2,3,6,7\}$	$3/2$	2	2
$\{1,2,4,5,7\}$	$7/3$	3	3
$\{1,2,4,5,8\}$	$7/3$	3	3
$\{1,3,5,7,9\}$	5	5	5
$\{1,2,3,4,5,6\}$	1	1	1
$\{1,2,3,4,5,7\}$	$5/3$	2	2
$\vdots$		$\vdots$	$\vdots$
$\{1,2,3,4,5,11\}$	$5/3$	2	2
$\{1,2,3,4,6,7\}$	$7/5$	2	2
$\{1,2,3,4,6,8\}$	$9/5$	2	3
$\{1,2,3,4,6,9\}$	$9/5$	2	3
$\{1,2,3,4,6,11\}$	$9/5$	3	3
$\{1,2,3,4,7,8\}$	$7/5$	2	2
$\{1,2,3,4,7,9\}$	$9/5$	2	3
$\{1,2,3,4,8,9\}$	$7/5$	2	2
$\{1,2,3,5,6,7\}$	2	2	3
$\{1,2,3,5,6,9\}$	2	3	3
$\{1,2,3,5,6,10\}$	2	3	3
$\{1,2,3,5,7,9\}$	$5/2$	3	4
$\{1,2,3,5,7,11\}$	$5/2$	3	4
$\{1,2,3,6,7,9\}$	2	3	3
$\{1,2,3,6,7,11\}$	2	3	3
$\{1,2,4,5,7,8\}$	$7/3$	3	3
$\{1,2,4,5,7,10\}$	3	4	4
$\{1,2,4,5,8,11\}$	3	4	4
$\{1,3,5,7,9,11\}$	6	6	6

Considering $depth(S)$ must be an integer, it seems at least one bound is always tight, and very often both are! Our bound

$$F(S) + 1 \geq m(S) + 2(dim_\alpha - 2)$$

reminds me of Wilf's conjecture, so let's go ahead and state that.

Conjecture 9 (Wilf) *Let $e(S)$ be the embedding dimension of S and $n(S)$ be the number of elements of S smaller than $F(S)$. Then*

$$F(S) + 1 \leq e(S)n(S)$$

I want to note here that the previously mentioned Shalom Eliahou wrote a paper on A Graph-Theoretic Approach to Wilf's Conjecture, and you know I'll have to check that out soon! His results prove Wilf's conjecture for 99.9999% of numerical semigroups - how did he get such a specific number? He defined a lot of similar stuff that we did here, but you can tell he's an expert at digging into the graph theory to extract good estimates.

As a summary, he associates a graph $G(S)$ to a numerical semigroup S so that the number of edges of that graph encodes the additive properties of S, via its Apery set X. We have an edge between $x, y \in X$ whenever $x + y \in X$ (and we allow for loops). He then uses vertex-matchings and weight-functions and graph decompositions to put bounds on the number of edges for this graph.

Eliahou actually uses depth here! I wonder if we'd have an analogue in terms of α-dimension, considering work a couple pages down showing the close relation of *depth* and α. But he actually defines a depth for each element in the semigroup (Which is maybe related to what I've been calling the partial genera?).

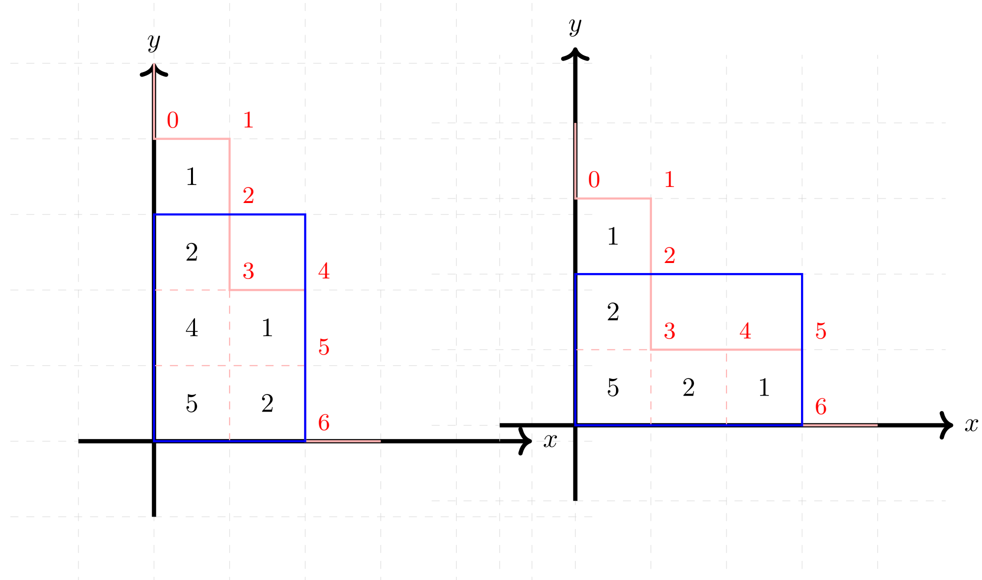
The total depth is $\tau(S)$, the sum of all depths. We let $\nu(S)$ be the size of the set of vertices in a maximal matching that don't contain a loop, and let $vm(S) = k$ be the size of maximum vertex-matching. then Eliahou shows

$$\tau(S) \geq (k(depth(S) - 1) + \nu(S))/2 + (n - k)$$

The next thing he does is show that if Wilf's Conjecture holds when $\tau(S) \leq 2depth(S) - 1$ (which intuitively would mean that there are few elements of S with large depth), then Wilf's conjecture holds for $n(S) \geq m(S)/3$. Again, employing some wonderful graphical analysis, he shows that Wilf's conjecture does indeed hold in these cases by the lower bound on $\tau(S)$, and therefore when $n(S) \geq m(S)/3$, which computationally (up to $g = 80$!!! I can barely get my computer to go past $g = 6$.) is true for over 99.9999% of numerical semigroups.

That was really cool! Moving on, if S is genus g, then we can rewrite $n(S) = F(S) - g + 1$, if that seems helpful. One thing of interest is that $n(S)$ is counting exactly the number of right-steps in our partition. As g is the height of our partition, $n(S)$ is the length.

Wilf's conjecture says that if we take a rectangle whose base is the base of the partition and whose height is the number of generators, then this area will always be greater than or equal to $F(S)+1$, which is the last element on our path.



Now the number of blocks in the partition is always at most $F(S)+1$, so perhaps a partition-inspired argument could come from moving any blocks outside the blue square into the square, and showing that this is always possible. This would be a slightly weaker result, depending on how much $\pi(S)$ and $F(S)+1$ differ.

If the conjecture were true, then by sandwhiching $F(S)$ between Wilf's upper bound and our Lemma's lower bound, we'd have

$$m(S) + 2(dim_\alpha(S) - 2) \leq e(S)n(S)$$

I wonder if we could prove this independently?

In terms of the partition, $dim_\alpha(S) - 2$ is the number of inner corners that aren't 0 or $F(S)+1$, while $m(S)$ represents how deep our first descent is. As we've said, $n(S)$ is the length of the bottom, but $e(S)$ doesn't quite have an interpretation I can see yet.

My next question is can we say anything about $e(S)$ in terms of $dim_\alpha(S)$? Well $\alpha(S)$ is generated by exactly the inner corners = the elements s so that $s-1 \notin S$. Keeping the L-semigroups in mind, we can keep $\alpha = 3$ and still increase g and $n(S)$ without bound. This also increases the Frobenius number and $e(S)$.

To be explicit,

$$g(L_{f,g}) = g$$

$$dim_\alpha(L_{f,g}) = 3$$

$$F(L_{f,g}) = f$$

and finally, since $f \leq 2g - 1$, we must have all elements $g, g+1, \ldots, f-1$ generators of S. So we get an inequality

$$e(S) \geq F(S) - g(S)$$

Which means that we can't expect a bound for $e(S)$ solely involving dim_α, which is kind of expected.

All of this gives me an idea for a sort of dual conjecture to Wilf's. The clearest duals here are $n(S)$ and g. To try to figure out a dual for $e(S)$, remember that $Irr(S)$ is the set of elements x in S so that there is no $s_1, s_2 \in S$ with $s_1 + s_2 = x$. This is a minimal generating set, so $e(S) = |Irr(S)|$.

We've been treating the dual concept as fragile gaps, those $y \in S^c$ with $s + y \in S$ for all non-zero $s \in S \cup \{y\}$, since

$$S \cup \{x\} \ numerical \ semigroup \Leftrightarrow x \ fragile$$

$$S - \{x\} \ numerical \ semigroup \Leftrightarrow x \ irreducible$$

So let's see where that takes us. First though, I want to look at *symmetric (and pseudo-symmetric) numerical semigroups.* I'll give the definition first, and then we'll see why they're named so.

Definition 6 A numerical semigroup S is called *symmetric* if $F(S) = 2g(S) - 1$. Note that $F(S)$ must be odd in this case. If $F(S)$ is even, then it is called *pseudo-symmetric* if $F(S) = 2g(S) - 2$.

Such semigroups pop up naturally when looking at numerical semigroups ordered by inclusion. In this poset, the irreducible elements are exactly the symmetric and pseudo-symmetric numerical semigroups.

To make sense of the name, let's go ahead and look at the two we just worked with, $\{1, 2, 5\}$ (symmetric), and $\{1, 2, 4, 5\}$ (not). I'll write the elements in S in blue and the gaps in red. We know all elements after $F(S)$ will be blue, so we'll stop there. Starting with $\{1, 2, 5\}$:

$$0, 1, 2, 3, 4, 5$$

There's symmetry! Flipping across the center, we get

$$5, 4, 3, 2, 1, 0$$

Swapping colors,

$$5, 4, 3, 2, 1, 0$$

And we see we get the same red-blue pattern as the original! And this is exactly why we call them symmetric. What happens for $\{1, 2, 4, 5\}$?

$$0, 1, 2, 3, 4, 5$$

Flipping across the center, we get

$$5, 4, 3, 2, 1, 0$$

Swapping colors,

$$5, 4, 3, 2, 1, 0$$

And we see we don't get the same pattern.

But we can do even more. If we look back at the partitions we got from these two (two pages up), you'll notice that $\{1, 2, 5\}$ is symmetric across the main diagonal, while $\{1, 2, 4, 5\}$ is not. And in general, this always holds!

$$S \textit{ symmetric} \longleftrightarrow \pi(S) \textit{ symmetric partition}$$

Just another reason why the partition is such a good visualization tool.

Now let's go ahead and look at fragile gaps - we'll write $fr(S)$ for the number of fragile gaps of S. The first thing we can notice is that if we fix a multiplicity $m(S)$, then any gap a of S larger than $F(S) - m(S)$ will be fragile if $2a \in S$. Recall that such gaps are called *fundamental gaps*.

Let's see a few examples of the differences between these two. I'll list a numerical semigroup and label fragile gaps blue, fundamental gaps red, green for both, and the others black.

$$\{1, 2, 3, 5, 7\}$$

$$\{1, 2, 4, 5, 8\}$$

$$\{1, 2, 3, 4, 5, 7, 9, 10, 11, 13\}$$

$$\{1, 2, 3, 4, 5, 6, 8, 9, 10, 11\}$$

10 Back to Partition Operations

Moving forward, here's one reason why this partition perspective is particularly helpful when thinking about the shift operator: it's literally just adding a block onto the first stack. Drawing a few pictures for each genus is very helpful to getting a feel for this. I wonder what a cyclic shift looks like? Let's start by interpreting some of the other operations we've looked at for partitions:

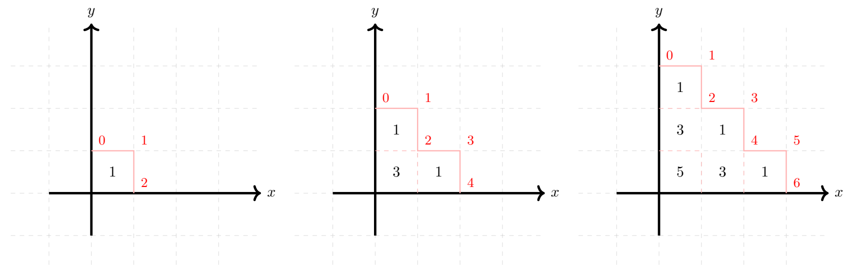
1. As mentioned before, the shift operator $\phi(S)$ is adding a single block to the first column.

2. Adding the Frobenius number (e.g. numerical semigroup tree) is the same as removing the bottom row of blocks.

3. Removing the multiplicity is removing a block from the second column and moving it to the first column.

So we see that such operations aren't particularly special. We could just as well add a block to any column and see if we get a numerical semigroup. We can move blocks around as we please. As long as we have a weakly decreasing path below the staircase partition, we have a chance to have a numerical semigroup.

Which means that we have *a lot* more operations on numerical semigroups than we've thought! We can take the partition as inspiration for new transformations. For example, what if we *cover* a numerical semigroup by dropping a block in each column (including the last empty one)?



Well this is just the opposite of adding the Frobenius number! So generating the numerical semigroup tree comes down to asking which partitions can have a bottom row added and still be a numerical semigroup.

Along with these transformations, we haven't addressed the most obvious question: What number n do we have $\lambda_S \vdash n$? (What we call n_S is what we were already calling $\pi(S)$. Oops! Keeping this anyway.) Denoting this as n_S, we can look at the generating function:

$$\mathcal{P}(g; x) = \sum_{S \in \mathcal{N}_g} x^{n_S}$$

to get a feel for this.

$$\mathcal{P}(0; x) = 1$$
$$\mathcal{P}(1; x) = x$$
$$\mathcal{P}(2; x) = x^2 + x^3$$
$$\mathcal{P}(3; x) = x^3 + x^4 + x^5 + x^6$$
$$\mathcal{P}(4; x) = x^4 + x^5 + x^6 + x^7 + 2x^8 + x^{10}$$

Two easy observations are

$$x^g || \mathcal{P}(g; x)$$
$$deg(\mathcal{P}(g; x)) = \binom{g+1}{2}$$

We also can look at whether $\mathcal{P}(g; -1)$ has any interpretation - i.e. is $\mathcal{P}(g; x)$ a Cyclic-Sieving Polynomial?. For the above five polynomials,

$$\mathcal{P}(0; -1) = 1$$

$$\mathcal{P}(1;-1) = -1$$
$$\mathcal{P}(2;-1) = 0$$
$$\mathcal{P}(3;-1) = 0$$
$$\mathcal{P}(4;-1) = 3$$

We definitely need to generate some more data but if this is a coincidence, it's a cool one:

$$\mathcal{P}(g;-1) = \sum_{\alpha=1}^{g+1} (-1)^\alpha n(g,\alpha)$$
$$g = 1, 2, 3, 4$$

Let's try with $g = 5$. We have

$$\sum_{\alpha=1}^{5+1} (-1)^\alpha n(5,\alpha) = -1 + 6 - 4 + 0 - 1 = 0$$

and

$$\mathcal{P}(5;x) = x^5 + x^6 + 2x^7 + 2x^8 + 4x^9 + x^{10} + x^{15}$$
$$\mathcal{P}(5;-1) = -4$$

So perhaps it was a coincidence!

I want to also say that just as we can compute α with the symmetric difference, we can compute n_S using the *partial genus* of a semigroup. The gaps split $\mathbb{N}$ into a number of intervals, so for each $s \in S$ the *partial genus* at s is $g_s(S) = |\{gaps > s\}|$. For example, with $\{1, 2, 3, 5, 6\}$, we have the partial genera (plural of genus):

$$g(S) = 5 \quad g_4(S) = 2$$

while for $\{1, 2, 3, 5, 7\}$, we have

$$g(S) = 5 \quad g_4(S) = 2 \quad g_6(S) = 1$$

For something like $\{1, 2, 3, 4, 8\}$, we get

$$g(S) = 5, g_5(S) = 1, g_6(S) = 1, g_7(S) = 1$$

And we can compute n_S as the sum of these partial genera.

```
{1, 2, 3, 4, 5}
pg: 5
pg: 0
pg: 0
pg: 0
----
{1, 2, 3, 4, 6}
pg: 5
pg: 1
pg: 0
pg: 0
----
{1, 2, 3, 4, 7}
pg: 5
pg: 1
pg: 1
pg: 0
----
{1, 2, 3, 4, 8}
pg: 5
pg: 1
pg: 1
pg: 1
{1, 2, 3, 4, 9}
pg: 5
pg: 1
pg: 1
pg: 1
pg: 1
----
{1, 2, 3, 5, 6}
pg: 5
pg: 2
pg: 0
pg: 0
----
{1, 2, 3, 5, 7}
pg: 5
pg: 2
pg: 1
pg: 0
----
{1, 2, 3, 5, 9}
pg: 5
pg: 2
pg: 1
pg: 1
pg: 1
{1, 2, 3, 6, 7}
pg: 5
pg: 2
pg: 2
pg: 0
----
{1, 2, 4, 5, 7}
pg: 5
pg: 3
pg: 1
pg: 0
----
{1, 2, 4, 5, 8}
pg: 5
pg: 3
pg: 1
pg: 1
----
{1, 3, 5, 7, 9}
pg: 5
pg: 4
pg: 3
pg: 2
pg: 1
```

And just as we sort by any statistic, we could of course try sorting numerical semigroups by n_S. For example,

$$n_S = 0 \Leftrightarrow S = \mathbb{N}$$

$$n_S = 1 \Leftrightarrow S = \{1\}^c$$

$$n_S = 2 \Leftrightarrow S = \{1, 2\}^c$$

And then for $n_S = 3$, we get $\mathcal{O}_3$ and $\{1, 3\}$. If we modify our n_S generating function by recording genus with y^g, then this gives us exactly the number we want if we look at a fixed x exponent. So

$$\overline{\mathcal{P}}(x, y) = \sum_{g=0}^{\infty} \sum_{S \in \mathcal{N}_g} x^{n_S} y^g$$

$$= 1 + xy + (x^2 + x^3)y^2 + (x^3 + x^4 + x^5 + x^6)y^3 + (x^4 + x^5 + x^6 + x^7 + 2x^8 + x^{10})y^4 +$$
$$+ (x^5 + x^6 + 2x^7 + 2x^8 + 4x^9 + x^{10} + x^{15})y^5 + \dots$$

And solving for x, we get

$$1 + (y)x + (y^2)x^2 + (y^2 + y^3)x^3 + (y^3 + y^4)x^4 + (y^3 + y^4 + y^5)x^5 +$$

$$+(y^3+y^4+y^5+y^6)x^6+\dots$$

And setting $y=1$ recovers the generating function for number of numerical semigroups with $n_S=k$ for a fixed k. And even though it's an infinite sum, we only need to calculate up to $g=k$ to get the exponent of x^k.

For the first few terms:

$$\overline{\mathcal{P}}(x,1)=1+x+x^2+2x^3+2x^4+3x^5+4x^6+\dots$$

The coefficients have a few possibilities on OEIS (E.g. A000009), but we definitely need to compute some more values. One of these days, I'll reinstall GAP on my computer and flesh out all of these computations for much larger genus.

Another thing I want to do is explore that $D_S(-1)$ a bit more. I'll list each semigroup with those values along with the other statistics we're interested in $(g(S),F(S),m(S),e(S),D_S(1))$.

S^c	S	$g(S)$	$F(S)$	$m(S)$	$e(S)$	$D_S(1)$	$D_S(-1)$
$\{\}$	$\langle 1\rangle$	0		1	1	3/2	1/2
$\{1\}$	$\langle 2,3\rangle$	1	1	2	2	2	1
$\{1,2\}$	$\langle 3,4,5\rangle$	2	2	3	3	2	0
$\{1,3\}$	$\langle 2,5\rangle$	2	3	2	2	3	1
$\{1,2,3\}$	$\langle 4,5,6,7\rangle$	3	3	4	4	2	1
$\{1,2,4\}$	$\langle 3,5,7\rangle$	3	4	3	3	3	0
$\{1,2,5\}$	$\langle 3,4\rangle$	3	5	3	2	3	0
$\{1,3,5\}$	$\langle 2,7\rangle$	3	5	2	2	4	1
$\{1,2,3,4\}$	$\langle 5,6,7,8,9\rangle$	4	4	5	5	2	0
$\{1,2,3,5\}$	$\langle 4,6,7,9\rangle$	4	5	4	4	3	1
$\{1,2,3,6\}$	$\langle 4,5,7\rangle$	4	6	4	3	3	1
$\{1,2,3,7\}$	$\langle 4,5,6\rangle$	4	7	4	3	3	1
$\{1,2,4,5\}$	$\langle 3,7,8\rangle$	4	5	3	3	3	1
$\{1,2,4,7\}$	$\langle 3,5\rangle$	4	7	3	2	4	0
$\{1,3,5,7\}$	$\langle 2,9\rangle$	4	7	2	2	5	1
$\{1,2,3,4,5\}$	$\langle 6,7,\ldots,11\rangle$	5	5	6	6	2	1
$\{1,2,3,4,6\}$	$\langle 5,7,8,9,11\rangle$	5	6	5	5	3	0
$\{1,2,3,4,7\}$	$\langle 5,6,8,9\rangle$	5	7	5	4	3	0
$\{1,2,3,4,8\}$	$\langle 5,6,7,9\rangle$	5	8	5	4	3	0
$\{1,2,3,4,9\}$	$\langle 5,6,7,8\rangle$	5	9	5	4	3	0
$\{1,2,3,5,6\}$	$\langle 4,7,9,10\rangle$	5	6	4	4	3	0
$\{1,2,3,5,7\}$	$\langle 4,6,9,11\rangle$	5	7	4	4	4	1
$\{1,2,3,5,9\}$	$\langle 4,6,7,11\rangle$	5	9	4	4	4	1
$\{1,2,3,6,7\}$	$\langle 4,5\rangle$	5	7	4	2	3	2
$\{1,2,4,5,7\}$	$\langle 3,8,10\rangle$	5	7	3	3	4	1
$\{1,2,4,5,8\}$	$\langle 3,7,8\rangle$	5	8	3	3	4	1
$\{1,3,5,7,9\}$	$\langle 2,11\rangle$	5	9	2	2	6	1
$\{1,2,3,4,5,6\}$	$\langle 7,\ldots,13\rangle$	6	6	7	7	2	0
$\{1,2,3,4,5,7\}$	$\langle 6,8,9,10,11,13\rangle$	6	7	6	6	3	1
$\{1,2,3,4,5,8\}$	$\langle 6,7,9,10,11\rangle$	6	8	6	5	3	1
$\{1,2,3,4,5,9\}$	$\langle 6,7,8,10,11\rangle$	6	9	6	5	3	1
$\{1,2,3,4,5,10\}$	$\langle 6,7,8,9,11\rangle$	6	10	6	5	3	1
$\{1,2,3,4,5,11\}$	$\langle 6,7,8,9,10\rangle$	6	11	6	5	3	1
$\{1,2,3,4,6,7\}$	$\langle 5,8,9,11,12\rangle$	6	7	5	5	3	1
$\{1,2,3,4,6,8\}$	$\langle 5,7,9,11\rangle$	6	8	5	4	4	0
$\{1,2,3,4,6,9\}$	$\langle 5,7,8,11\rangle$	6	9	5	4	4	0
$\{1,2,3,4,6,11\}$	$\langle 5,7,8,9,11\rangle$	6	11	5	5	4	0
$\{1,2,3,4,7,8\}$	$\langle 5,6,9\rangle$	6	8	5	3	3	-1
$\{1,2,3,4,7,9\}$	$\langle 5,6,8\rangle$	6	9	5	3	4	0
$\{1,2,3,4,8,9\}$	$\langle 5,6,7\rangle$	6	9	5	3	3	1
$\{1,2,3,5,6,7\}$	$\langle 4,9,10,11\rangle$	6	7	4	4	3	1
$\{1,2,3,5,6,9\}$	$\langle 4,7,9,10\rangle$	6	9	4	4	4	0
$\{1,2,3,5,6,10\}$	$\langle 4,7,9\rangle$	6	10	4	3	4	0
$\{1,2,3,5,7,9\}$	$\langle 4,6,11,13\rangle$	6	9	4	4	5	1
$\{1,2,3,5,7,11\}$	$\langle 4,6,9,13\rangle$	6	11	4	4	5	1
$\{1,2,3,6,7,11\}$	$\langle 4,5,11,13\rangle$	6	11	4	4	4	2
$\{1,2,4,5,7,8\}$	$\langle 3,10,11\rangle$	6	8	3	3	4	0
$\{1,2,4,5,7,10\}$	$\langle 3,8\rangle$	6	10	3	2	5	1
$\{1,2,4,5,8,11\}$	$\langle 3,7\rangle$	6	11	3	2	5	1
$\{1,3,5,7,9,11\}$	$\langle 2,13\rangle$	6	11	2	2	7	1

This at least cleared up why $D_S(-1)$ is always an integer:

Proposition 6 *For all nontrivial numerical semigroups S, both $D_S(1)$ and $D_S(-1)$ are integers.*

Proof 6 *First,the symmetric difference of two numerical semigroups has even size, since we can pair up elements that are in and not in S (and all that Hamming distance stuff!). Also, the symmetric difference $S \triangle (1+S)$ is the same as $S^c \triangle (1+S^c) \cup \{0\}$, so we get an even size + $\{0\}$, which is odd. This shows $D_S(1)$ is an integer.*

To see why $D_S(-1)$ is an integer, we can partition the set $S \triangle (1+S)$ into the even elements E and the odd elements O. Since the total size is odd, one of these sets must have odd size. If it's the even set, then $1+|E|-|O|$ is an even number minus an even number, and therefore even. If it's the odd set, then we similarly have $|E|-|O|$ odd, so $1+|E|-|O|$ is even. Therefore, $D_S(-1)$ is an integer.

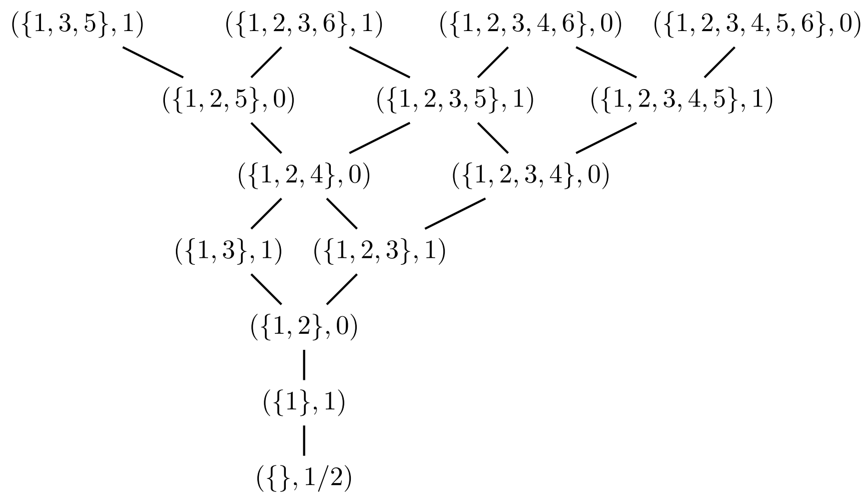
Let's call a numerical semigroup *corner-balanced* if $D_S(-1) = 0$. Another way to characterize this would be saying that $S \triangle (1+S)$ contains exactly one more odd number than even numbers. I'd love a way to decide without explicit calculation whether a semigroup is corner-balanced. We can of course see a few patterns:

$$D_{\mathcal{O}_g}(-1) = \begin{cases} 1 & g \text{ odd} \\ 0 & g \text{ even} \end{cases}$$

since $S \triangle (1+S) = \{0, 1, g+1\}$.

Remember we talked about the shifting operation as adding a single block onto the first column. This, by its name, shifts all elements *backwards* by 1 along the path of the partition. In general, we can add a block to any inner corner and turn it into an outer corner, which is equivalent to taking the first element in a continuous interval interval of S and swapping it out with the gap before it. If we do this with 0 (the normal shift operator), then it increases the genus. If we add a block to the last spot, we increase the Frobenius number. If we do it with any other inner corner, it doesn't change either.

I think to get a better feel for this, let's draw a poset where numerical semigroups $S < T$ if $\pi(T)$ comes from $\pi(S)$ by adding a single block to an inner corner. Let's call this $\mathcal{B}$ for block. I'll list the numerical semigroups and its $D_S(-1)$ value, to see if there is a connection.



Let's get the rank generating function. A helpful tool here is Macaulay2, which has great resources for working in commutative algebra, including operations on posets.

$$R_{\mathcal{B}}(x) = 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + \dots$$

Oh yea, this will be exactly the same as $\mathcal{P}(x, 1)$, as we previously defined.

Conjecture 10 *The coefficients of $R_{\mathcal{B}}(x)$ are weakly increasing.*

Let's get more specific with the block adding. If we have an inner corner $s \neq 0$, then adding a block there "pops" the corner out, which in terms of the numerical semigroup, swaps out s and $s - 1$. Now we have a couple of cases. If $s - 1$ was already a corner, then it's no longer a corner. If it wasn't a corner, it now is. The same goes for $s + 1$

Let's call an inner corner a *left-deep corner* if it isn't 0 and it isn't right after another corner (meaning $s-1, s-2 \in S^c$). We'll call it a *right-deep corner* if it isn't 0 and isn't right before another corner (meaning $s + 1, s + 2 \in S$). And naturally a *deep corner* will be an inner corner that is both left-deep and right-deep. Here's an example of the three:

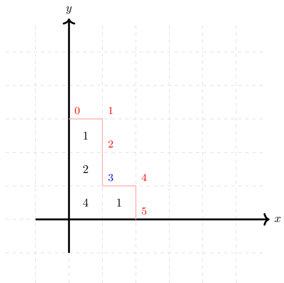


Figure 13: Left-deep corner

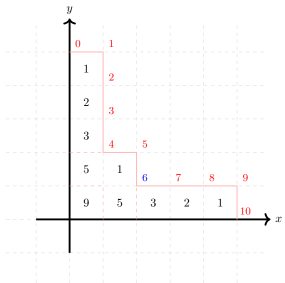


Figure 14: Right-deep corner

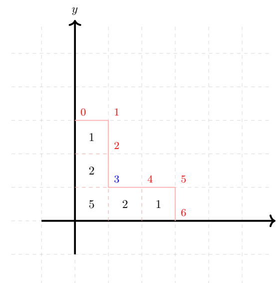


Figure 15: Deep corner

Let's denote $\alpha_\ell(S), \alpha_r(S)$, and $\alpha_d(S)$ for the number of left-deep, right-deep, and deep, respectively, corners of S. An immediate equation relating these is pretty much the venn diagram (inclusion-exclusion):

$$\alpha(S) = 2 + (\alpha_\ell(S) + \alpha_r(S) - \alpha_d(S))$$

For the above examples, we have

$$\alpha_\ell(\{1,2,4\}) = 1 \quad \alpha_r(\{1,2,4\}) = 0 \quad \alpha_d(\{1,2,4\}) = 0 \quad \alpha(\{1,2,4\}) = 3$$

$$\alpha_\ell(\{1,2,3,5,9\}) = 1 \quad \alpha_r(\{1,2,3,5,9\}) = 1 \quad \alpha_d(\{1,2,4\}) = 0 \quad \alpha(\{1,2,4\}) = 4$$

$$\alpha_\ell(\{1,2,5\}) = 1 \quad \alpha_r(\{1,2,5\}) = 1 \quad \alpha_d(\{1,2,5\}) = 1 \quad \alpha(\{1,2,4\}) = 3$$

Now let's think about what adding a block to these different types of corners does. Let S be the numerical semigroup and S^d, S^ℓ, S^r be the set with a deep, left-deep, and right-deep corner block added (not necessarily a numerical semigroup).

First, if we have a deep corner, then adding a block will add 2 inner corners and add 1 outer corner, while removing 1 inner corner. This means this increases α by 1 and our set of corners increases from A to $A \cup \{s-1, s+1\}$. So our corner-generating function will go

$$D_S(x) = \frac{1}{2}\left(1 + \sum_{n\in A} x^n\right) \longrightarrow D_{S^d}(x) = \frac{1}{2}\left(1 + x^{s-1} + x^{s+1} + \sum_{n\in A} x^n\right)$$

which means

$$D_{S^d}(x) = \frac{x^{s-1} + x^{s+1}}{2} + D_S(x)$$

Setting $x = -1$ gives

$$D_{S^d}(-1) = \frac{(-1)^{s-1} + (-1)^{s+1}}{2} + D_S(-1) = (-1)^{s-1} + D_S(-1)$$

Awesome! So this is how a deep corner being added affects $D_S(-1)$. We can see an example of this from $\{1,2,5\} \to \{1,3,5\}$, which sends $D_{\{1,2,5\}}(-1) = 0$ to $D_{\{1,3,5\}}(-1) = 1$, since the corner we added a block to was 3, which is odd.

Ok, so now what if we add a left-deep block (that is not deep)? Then s goes from inner corner to outer corner, while $s-1$ becomes a new inner corner, and $s+1$ is no longer a corner. This doesn't change α, but it means we go from a set A of corners to $A \cup \{s-1\} - \{s+1\}$, and so

$$D_{S^\ell}(x) = \frac{x^{s-1} - x^{s+1}}{2} + D_S(x)$$

Setting $x = -1$ gives

$$D_{S^\ell}(-1) = \frac{(-1)^{s-1} - (-1)^{s+1}}{2} + D_S(-1) = D_S(-1)$$

Great, this is exactly the characterization I wanted. Adding a left-deep corner doesn't change $D_S(-1)$. Let's do right-deep for completeness, despite the symmetry.

Adding a right-deep (non-deep) corner s will take A to $A \cup \{s+1\} - \{s-1\}$, so

$$D_{S^r}(x) = \frac{x^{s+1} - x^{s-1}}{2} + D_S(x)$$

Setting $x = -1$ gives

$$D_{S^r}(-1) = \frac{(-1)^{s+1} - (-1)^{s-1}}{2} + D_S(-1) = D_S(-1)$$

Adding a block on the last section (increasing $F(S)$ by 1) works the same as any other right-deep (or deep) corner. The oddest one is adding a block on the first column, because we relabel starting at 0, meaning all elements are shifted backwards by 1. Since we're keeping the partition shape the same, this means all corners A larger than 1 are shifted forward by 1, and we keep $0, 1$ at the beginning. So A goes to $(1 + A) - \{2\} \cup \{0\}$.

Recall $\phi(S) = (1 + S) \cup \{0\}$. So

$$D_{\phi(S)}(x) = \frac{1}{2}\left(1 + 1 - x^2 + \sum_{n \in A} x^{n+1}\right) = \frac{1}{2}\left(2 - x^2 + x\sum_{n \in A} x^n\right)$$

Writing the sum in terms of $D_S(x)$ gives

$$= xD_S(x) - \frac{(x+2)(x-1)}{2}$$

Setting $x = -1$ gives

$$D_{\phi(S)}(-1) = (-1)D_S(-1) + 1$$

That's fun! It explains the alternating value for $\mathcal{O}_g$ in another way. We start at 1 when $g = 1$ and then go to $(-1)(1) + 1 = 0$, then to $(-1)0 + 1 = 1$, and back and forth.

We saw before that repeated shifting will end up in a chain of numerical semigroups, so what happens to $D_{\phi^k(S)}(-1)$ as we interate ϕ? Well

$$
\begin{aligned}
D_{\phi^k(S)}(-1) &= (-1)D_{\phi^{k-1}(S)}(-1) + 1 \\
&= (-1)((-1)D_{\phi^{k-2}(S)}(-1) + 1) + 1 \\
&= D_{\phi^{k-2}(S)}(-1) \\
&\vdots \\
&= \begin{cases} D_S(-1) & k \textit{ even} \\ (-1)D_S(-1) + 1 & k \textit{ odd} \end{cases}
\end{aligned}
$$

So maybe what we should actually be looking at here is what $D_S(-1)$ tells us about whether S is shifted or unshifted. After all, we do still have that conjecture way back that the number of shifted and unshifted numerical semigroups is the same (± 1 if $N(g)$ odd).

For example, if $D_S(-1) = 0$ or 1, then its shifts will alternate between 0 and 1. But if $D_S(-1) = 2$, then $D_{\phi(S)}(-1) = -1$, and we alternate between those two. Does this explain why that one negative value popped up? Recall

$$D_{\{1,2,3,4,7,8\}}(-1) = -1$$

was the first negative value, but now I see it only came *after* the first 2 appeared,

$$D_{\{1,2,3,6,7\}}(-1) = 2$$

And they are connected by shifts!

$$\phi(\{1, 2, 3, 6, 7\}) = \{1, 2, 3, 4, 7, 8\}$$

And further, the fact that S is the first numerical semigroup with $D_S(-1) = 2 \neq 1, 0$ means that S must be unshifted! After that, any numerical semigroup S with $D_S(-1) \neq -1, 0, 1, 2$ must be unshifted. And so on.

This very cool. I'm going to copy that shifted numerical semigroup forest, along with the value of $D_S(-1)$ listed on the arrow coming out of S.

$$
\begin{array}{llllll}
\{1\} \xrightarrow{1} & \{1,2\} \xrightarrow{0} & \{1,2,3\} \xrightarrow{1} & \{1,2,3,4\} \xrightarrow{0} & \{1,2,3,4,5\} \xrightarrow{1} & \{1,2,3,4,5,6\} \xrightarrow{0} \\
 & \{1,3\} \xrightarrow{1} & \{1,2,4\} \xrightarrow{0} & \{1,2,3,5\} \xrightarrow{1} & \{1,2,3,4,6\} \xrightarrow{0} & \{1,2,3,4,5,7\} \xrightarrow{1} \\
 & & \{1,2,5\} \xrightarrow{0} & \{1,2,3,6\} \xrightarrow{1} & \{1,2,3,4,7\} \xrightarrow{0} & \{1,2,3,4,5,8\} \xrightarrow{1} \\
 & & \{1,3,5\} \xrightarrow{1} & \{1,2,4,6\} \xrightarrow{0} & \{1,2,3,5,7\} \xrightarrow{1} & \{1,2,3,4,6,8\} \xrightarrow{0} \\
 & & & \{1,2,3,7\} \xrightarrow{1} & \{1,2,3,4,8\} \xrightarrow{0} & \{1,2,3,4,5,9\} \xrightarrow{1} \\
 & & & \{1,2,4,5\} \xrightarrow{1} & \{1,2,3,5,6\} \xrightarrow{0} & \{1,2,3,4,6,7\} \xrightarrow{1} \\
 & & & \{1,2,4,7\} \xrightarrow{0} & \{1,2,3,5,8\} \xrightarrow{1} & \{1,2,3,4,6,9\} \xrightarrow{0} \\
 & & & \{1,3,5,7\} \xrightarrow{1} & \{1,2,4,6,8\} \xrightarrow{0} & \{1,2,3,5,7,9\} \xrightarrow{1} \\
 & & & & \{1,2,3,4,9\} \xrightarrow{0} & \{1,2,3,4,5,10\} \xrightarrow{1} \\
 & & & & \{1,2,3,5,9\} \xrightarrow{1} & \{1,2,3,4,6,10\} \xrightarrow{0} \\
 & & & & \{1,2,3,6,7\} \xrightarrow{2} & \{1,2,3,4,7,8\} \xrightarrow{-1} \\
 & & & & \{1,2,4,5,7\} \xrightarrow{1} & \{1,2,3,5,6,8\} \xrightarrow{0} \\
 & & & & \{1,2,4,5,8\} \xrightarrow{1} & \{1,2,3,5,6,9\} \xrightarrow{0} \\
 & & & & \{1,3,5,7,9\} \xrightarrow{1} & \{1,2,4,6,8,10\} \xrightarrow{0} \\
 & & & & & \{1,2,3,4,5,11\} \xrightarrow{1} \\
 & & & & & \{1,2,3,4,6,11\} \xrightarrow{0} \\
 & & & & & \{1,2,3,4,7,9\} \xrightarrow{0} \\
 & & & & & \{1,2,3,4,8,9\} \xrightarrow{1} \\
 & & & & & \{1,2,3,5,6,7\} \xrightarrow{1} \\
 & & & & & \{1,2,3,5,6,10\} \xrightarrow{0} \\
 & & & & & \{1,2,3,5,7,11\} \xrightarrow{1} \\
 & & & & & \{1,2,3,6,7,11\} \xrightarrow{1} \\
 & & & & & \{1,2,4,5,7,8\} \xrightarrow{0} \\
 & & & & & \{1,2,4,5,7,10\} \xrightarrow{1} \\
 & & & & & \{1,2,4,5,8,11\} \xrightarrow{1} \\
 & & & & & \{1,3,5,7,9,11\} \xrightarrow{1}
\end{array}
$$

That's wonderful! I wonder if there is periodicity for other roots of unity? We'll think about that later. Let's add up the columns first:

$$1, 1, 2, 5, 9, 13$$

This does not appear in the OEIS. Notice that

$$1 \leq 1 \leq 1$$

$$1 \leq 1 \leq 2$$

$$2 \leq 2 \leq 4$$

$$4 \leq 5 \leq 7$$

$$7 \leq 9 \leq 12$$

$$12 \leq 13 \leq 23$$

I want to say more about the $D_S(-1)$ values, but at the moment, I'll just leave a conjecture.

Conjecture 11 *For all $g \geq 1$, we have*

$$N(g-1) \leq \sum_{S \in \overline{\mathcal{N}}_g} D_S(-1) \leq N(g)$$

And of course (as the point has been with many of the things we've done in Part 2), this would imply $N(g-1) \leq N(g)$.

There are definitely struggles with having such a large document - repeating notation! But I guess we'll hope this is unused and call

$$\mathcal{D}(g) = \sum_{S \in \overline{\mathcal{N}}_g} D_S(-1)$$

Let's recall some ideas and notation from page 40ish. We're using $N_0(g)$ to represent the number of unshifted numerical semigroups of genus g, and conjecture that $N_0(g) = \lceil N(g)/2 \rceil$. We use $\overline{N}(g)$ to be the number of sets in the g^{th} column. So

$$N(g) \leq \overline{N}(g)$$

and the fact that $\overline{N}(g-1) \leq \overline{N}(g)$ is obvious from the definition of the forest we made (i.e. shift each set in previous column and then add some more). Which gives

$$\overline{N}(g) = \overline{N}(g-1) + N_0(g),$$

which recursively gives

$$\overline{N}(g) = \sum_{k=1}^{g} N_0(k)$$

Now let's look at a recursion for $\mathcal{D}(g)$. We'll split notation already and call $\mathcal{D}_0(g)$ the sum over unshifted numerical semigroups of genus g. Then

$$\mathcal{D}(g) = \mathcal{D}_0(g) + \sum_{S \in \overline{\mathcal{N}}_{g-1}} D_{\phi(S)}(-1)$$

$$= \mathcal{D}_0(g) + \sum_{S \in \overline{\mathcal{N}}_{g-1}} (-1)\mathcal{D}_S(-1) + 1$$

$$= \mathcal{D}_0(g) + \overline{N}(g-1) - \sum_{S \in \overline{\mathcal{N}}_{g-1}} \mathcal{D}_S(-1)$$

which finally gives

$$\mathcal{D}(g) = \mathcal{D}_0(g) + \overline{N}(g-1) - \mathcal{D}(g-1)$$

So what would an inductive argument look like? Well if

$$N(g-2) \leq \mathcal{D}(g-1) \leq N(g-1),$$

then

$$-N(g-1) \leq -\mathcal{D}(g-1) \leq -N(g-2),$$

so

$$\mathcal{D}_0(g) + \overline{N}(g-1) - N(g-1) \leq \mathcal{D}(g) \leq \mathcal{D}_0(g) + \overline{N}(g-1) - N(g-2).$$

From this, we want to deduce

$$N(g-1) \leq \mathcal{D}(g) \leq N(g)$$

Can we just straight up sandwich the bounds? Let's go ahead and list out $\mathcal{D}_0(g)$ for $g = 1, 2, \ldots$

$$1, 1, 1, 3, 6, 8$$

There are a few possibilities on OEIS, so I'll have to compute more terms later.

Let's start with the left-hand side and $N(g-1)$, starting with $g=2$.

$$\mathcal{D}_0(g) + \overline{N}(g-1) - N(g-1): \ 1, 1, 3, 7, 10$$
$$N(g-1): \ 1, 2, 4, 7, 12$$

Here are the central values, again starting with $g=2$.

$$\mathcal{D}(g): \ 1, 2, 5, 9, 13$$

And now we'll do the right-hand side, starting with $g=2$.

$$\mathcal{D}_0(g) + \overline{N}(g-1) - N(g-2): \ 1, 2, 5, 10, 15$$
$$N(g): \ 2, 4, 7, 12, 23$$

So it appears that the upper bound may indeed be bound above by $N(g)$, which is a definite avenue to explore for a proof. But $N(g-1)$ is often larger than the lower bound, so that won't work there. And it seems like $\mathcal{D}(g)$ is much closer to $N(g-1)$ than $N(g)$, so that kind of makes sense. It's also not too surprising if we had to separate the proof into the upper and lower bound separately (if we could even prove this).

In fact, it's such a tight bound that $\overline{N}(g-1) > \mathcal{D}(g)$ for a few g values already. But it's possible that the conjectured lower bound would fail if we calculate a few more values of g.

NOTE: I definitely need to calculate higher values to see if these conjectures are even true for relatively low values of g

As we unwound a recursion to find

$$\overline{N}(g) = \sum_{k=1}^{g} N_0(k),$$

let's do the same with $\mathcal{D}(g)$.

$$\begin{aligned}\mathcal{D}(g) &= \mathcal{D}_0(g) + \overline{N}(g-1) - \mathcal{D}(g-1) \\ &= \mathcal{D}_0(g) + \overline{N}(g-1) - (\mathcal{D}_0(g-1) + \overline{N}(g-2) - \mathcal{D}(g-2)) \\ &= \mathcal{D}_0(g) + \overline{N}(g-1) - \mathcal{D}_0(g-1) - \overline{N}(g-2) + \mathcal{D}(g-2)\end{aligned}$$

$$------$$

$$= \mathcal{D}_0(g) + \overline{N}(g-1) - \mathcal{D}_0(g-1) - \overline{N}(g-2) + \mathcal{D}_0(g-2) + \overline{N}(g-3) - \mathcal{D}(g-3)$$

Continuing to iterate gives

$$\mathcal{D}(g) = \sum_{k=1}^{g-1} (-1)^{g-1-k} (\mathcal{D}_0(k+1) + \overline{N}(k))$$

Let's isolate the $\overline{N}(k)$ portion and see if replacing it with the sum above gives anything.

$$\begin{aligned}(-1)^{g-1}\sum_{k=1}^{g-1}(-1)^k\overline{N}(k) &= (-1)^{g-1}\sum_{k=1}^{g-1}(-1)^k\sum_{j=1}^{k}N_0(j)\\ &= (-1)^{g-1}\sum_{j=1}^{g-1}\sum_{k=j}^{g-1}(-1)^j N_0(j)\\ &= (-1)^{g-1}\sum_{j=1}^{g-1}(-1)^j(g-j)N_0(j)\end{aligned}$$

I'm not sure if there's much more to do right now. But remember column j (starting from 0) must be less than $g-j$ for the partition to sit under the staircase partition. So we move along the columns and form a box the tallest box whose length is $N_0(j)$, alternating the sum...For now, I'll leave that there.

I want to get back to $D_S(x)$ in general. We have

$$D_{\phi(S)}(x) = xD_S(x) - \frac{(x+2)(x-1)}{2}$$

$$\begin{aligned}D_{S^\ell}(x) &= D_S(x) + \frac{x^{s-1}-x^{s+1}}{2}\\ D_{S^d}(x) &= D_S(x) + \frac{x^{s-1}+x^{s+1}}{2}\\ D_{S^r}(x) &= D_S(x) + \frac{x^{s+1}-x^{s-1}}{2}\end{aligned}$$

Looking back at the shifting forest, although $\{1,3\}$ is unshifted, it can be obtained from $\{1,2\}$ by adding a block. That is,

$$\{1,3\} = \{1,2\}_3^d$$

For $g=3$, we have the chain

$$\begin{aligned}\{1,2,4\} &= \{1,2,3\}_4^d\\ \{1,2,5\} &= \{1,2,4\}_5^r\\ \{1,3,5\} &= \{1,2,5\}_3^d\end{aligned}$$

I don't want to get too excited, *but* if we fix a genus g and look at $\mathcal{B}_g$, the poset formed by adding blocks, then we have $\mathcal{O}_g = \hat{0}$ and $\{1,3,\ldots,2g-1\} = \hat{1}$, like we've mentioned before "should" be the case!

Let's define

$$\mathcal{D}(g;x) = \sum_{S\in\overline{\mathcal{N}}_g} D_S(x)$$

$$= \sum_{S\in\overline{\mathcal{N}}_g} \frac{1}{2}\left(1 + \sum_{n\in S\triangle(1+S)} x^n\right)$$

$$= \frac{1}{2}\left(\overline{N}(g) + \sum_{S\in\overline{\mathcal{N}}_g} \sum_{n\in S\triangle(1+S)} x^n\right)$$

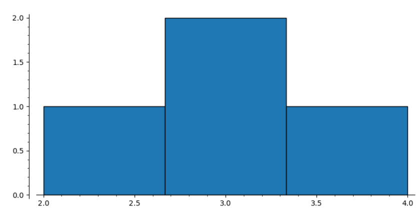
$$= \frac{1}{2}\left(\overline{N}(g) + \sum_{n=0}^{2g-1} C_g(n)x^n\right)$$

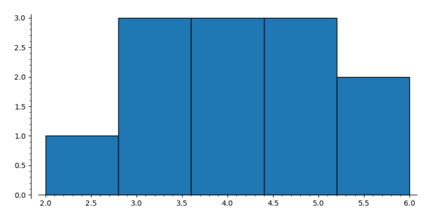
where $C_g(n)$ is the number of sets in $\overline{\mathcal{N}}_g$ that have n as a corner.

So $\mathcal{D}(g)$ as we previously defined is equal to $\mathcal{D}(g;-1)$. What is $\mathcal{D}(g;1)$? Well $D_S(1)$ is the number of inner corners of S, so $\mathcal{D}(g;1)$ would be the total number of inner corners for all sets in $\overline{\mathcal{N}}_g$. Ok.

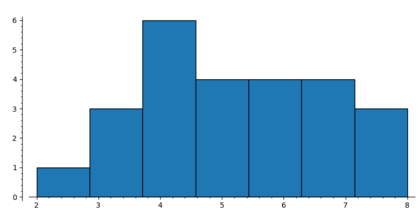
Can we better describe the coefficients $C_g(n)$? They're fairly natural to look at. Since we're using the corners of the partition to study the numerical semigroups, we'd definitely also want to look at the distribution of those corners.

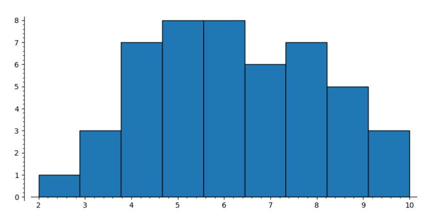
Naturally we have $C_g(0) = C_g(1) = \overline{N}(g)$. So let's plot a histogram of the values for various g. First, $g = 2$ and 3.



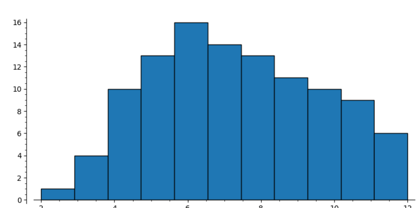


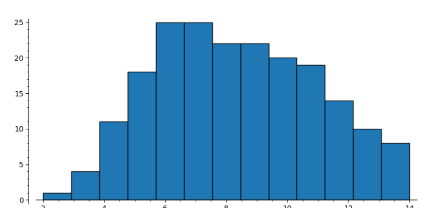
And $g = 4, 5$



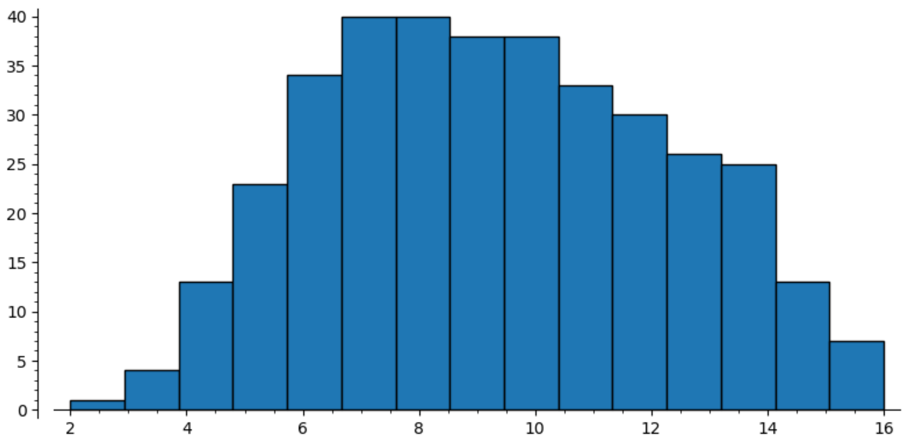


And $g = 6, 7$





And finally $g = 8$. Again, we really need to reinstall GAP...



In the interval $[2, 2g]$, it seems the distribution has a slight right skew, with mode g, which I guess also makes sense. Let's list the maximal counts for $g = 2, 3, \ldots$:

$$2, 3, 6, 8, 16, 25, 40$$

This does not appear in the OEIS. The results from a partial sequence are interesting too!

Let's list a few of the values at $x = 1, -1$.

$$\mathcal{D}(2; x) = \frac{1}{2}(4 + 2x + x^2 + 2x^3 + x^4)$$

$$\mathcal{D}(2; 1) = 5 \qquad \mathcal{D}(2; -1) = 1$$

$$-------------------$$

$$\mathcal{D}(3; x) = \frac{1}{2}(8 + 4x + x^2 + 3x^3 + 3x^4 + 3x^5 + 2x^6)$$

$$\mathcal{D}(3; 1) = 12 \qquad \mathcal{D}(3; -1) = 2$$

$$-------------------$$

$$\mathcal{D}(4; 1) = 27 \qquad \mathcal{D}(4; -1) = 5$$

$$-------------------$$

$$\mathcal{D}(5; 1) = 51 \qquad \mathcal{D}(5; -1) = 9$$

$$-------------------$$

$$\mathcal{D}(6; 1) = 102 \qquad \mathcal{D}(6; -1) = 14$$

$$-------------------$$

Just as we conjectured that

$$N(g - 1) \le \mathcal{D}(g; -1) \le N(g),$$

it also seems that

$$\mathcal{D}(g;1) \geq N(g+2)$$

We *might* conjecture an upper bound of $N(g+3)$, but I'm not convinced $\mathcal{D}(g;1)$ won't outgrow it - it seems to have exponential growth with base around 2.

11 NS Recap

So in the last few sections, we talked about *a lot* of possibilities to look into in terms of studying numerical semigroups via their partitions. We've defined (and recorded others' definitions of) many different statistics related to numerical semigroups. For any stat st, we can form the statistic generating function

$$ST(x) = \sum_{S \in \mathcal{N}} x^{st(S)}$$

We could even combine statistics in a multi-variate generating function if we want to see how they interact. For example, if you want to see how the Frobenius number and multiplicity interact, you could look at the generating function

$$ST(x,y) = \sum_{S \in \mathcal{N}} x^{F(S)} y^{m(S)}$$

If we have the general viewpoint of trying to prove $N(g) \geq N(g-1)$, then we probably want to see how our statistic splits up numerical semigroups of a fixed genus:

$$ST(x,y) = \sum_{S \in \mathcal{N}} x^{st(S)} y^{g(S)}$$

And then we have

$$ST(1,y) = \sum_{g=0}^{\infty} N(g) y^g$$

while

$$ST(x,1) = \sum_{k=0}^{\infty} N_{st}(k) x^k$$

where $N_{st}(k)$ is the number of numerical semigroups with $st(S) = k$. Let's go ahead and list out every numerical semigroup statistic we've looked at so far:

11.1 Genus and Partial Genus

The genus of a numerical semigroup S is the size of S^c, the set of gaps. This is generally denoted by $g(S)$ and the number of numerical semigroups of genus g is $N(g)$.

The partial genus at $s \in S$ is $g_s(S) = |\{gaps > s\}|$. The usual genus is $g_0(S)$. The sequence of partial genera determines a numerical semigroup.

11.2 Frobenius Number and Conductor

The Frobenius number of a numerical semigroup S is the largest gap. This is generally denoted by $F(S)$. Closely related is the conductor of S, which is the smallest element of S larger than every gap of S, i.e. $c(S) = F(S) + 1$.
Resources:

1. Counting numerical semigroups by Frobenius number, multiplicity, and depth by Sean Li.

11.3 Multiplicity

The Multiplicity of a numerical semigroup S is the smallest non-zero element of S.
Resources:

1. Counting numerical semigroups by Frobenius number, multiplicity, and depth by Sean Li.

11.4 Depth

The depth of a numerical semigroup S is defined to be $depth(S) = \lceil c(S)/m(S) \rceil$.
Resources:

1. Counting numerical semigroups by Frobenius number, multiplicity, and depth by Sean Li.

11.5 Embedding Dimension

The Embedding Dimension of a numerical semigroup S is the size of a minimum generating set of S.

12 Starting Fresh

Restricting the height of a partition puts a limit on the genus of numerical semigroups. Restricting the length puts a limit on the number of intervals S cuts $\mathbb{N}$ into, which is also partially restricts the alpha-dimension. But to explicitly restrict the alpha dimension is like taking a hyperbola and forcing our partition to be under it. When we have no α restriction, this line straightens out to the diagonal $(0, g) - (g, 0)$.

For $\alpha = 1$, this is a very steep hyperbola (almost at both axes). For $\alpha = 2$, it's wide enough to allow a single column or row but nothing else. For $\alpha = 3$, it widens enough to allow both a first row and column, or a first column and second column. (e.g. $\{1, 2, 5\}$ has a row and column but $\{1, 2, 4, 5\}$ has a column of height 4 and then one of height 2, but still has $\alpha = 3$.

As such, I think this is a fairly different way to order numerical semigroups, so I certainly want to try to learn more about the values $n(g,\alpha)$. Using the examples in the last paragraph, we have

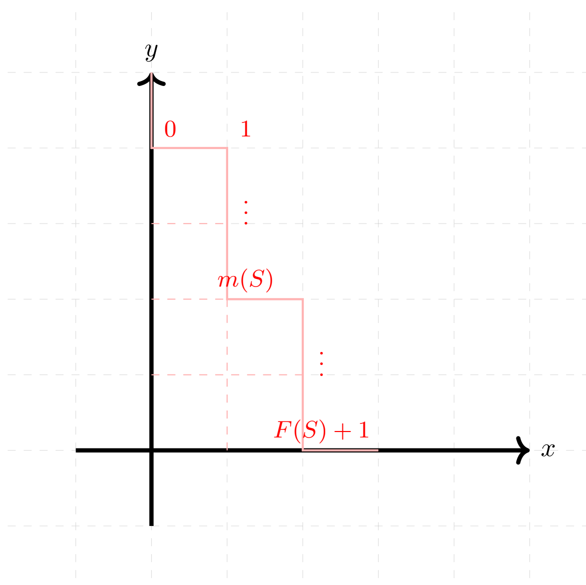
$$n(g,1) = \begin{cases} 1 & g = 0 \\ 0 & g > 0 \end{cases}$$

from $\{\}$. We have

$$n(g,2) = \begin{cases} 0 & g = 0 \\ 1 & g > 0 \end{cases}$$

from $\mathcal{O}_g$.

For $\alpha = 3$, we have the $L_{f,g}$ semigroups (for which $f = g+1, \ldots, 2g-1$, so we have $g-1$ of them) and the numerical semigroups made up of two columns. Which of those are actually numerical semigroups?



Such a partition would give a complement set of the form

$$\{1, 2, \ldots, m-1, m+1, \ldots, g+1\}.$$

To not double count, we can assume $m(S) < g$. This will only fail to be a numerical semigroup if $2m \in S^c$, which means $2m \leq g+1$, so we get $m \leq (g+1)/2$. Therefore, the remaining choices

$$g - 1 - \frac{g+1}{2} = \frac{g-3}{2}$$

will give a numerical semigroup. This finally gives

$$n(g,3) = (g-1) + \left\lceil \frac{g-3}{2} \right\rceil$$

but that is low for $g = 6$.

And I see why! We were too restricted. It's not just the two columns. The condition is $\alpha = 3$ if and only if $\pi(S) = g + k + k + \cdots + k$ for some non-zero k. The almost-ordinary semigroups $L_{f,g}$ have $k = 1$. But consider $S = \{1, 2, 3, 4, 7, 8\}$, which has partition $\pi(S) = 6 + 2 + 2$, coming from the partial genera $g_0(S) = 6, g_5(S) = 2, g_6(S) = 2$. This has $\alpha = 3$.

Let's try an example. Fixing $g = 5$, we're working with $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, then choosing $f = 7$ would force $t = 2$, so we could remove 56 or 45, but once we get to 34, we run into the issue that $3+4 = 7$. And this is because $2(3) < 7$.

If we choose $f = 9$, then we get $t = 4$, and we could remove 5678 to get $\{1, 2, 3, 4, 9\}$ and moving any lower causes an issue because $2(4) = 8$.

If we choose $f = 8$, then we have $t = 3$, and we can remove 567 to get $\{1, 2, 3, 4, 8\}$ but 456 gives a problem because $2(4) = 8$, like before.

I could work out a general pattern (depending on even/odd), but I feel very uncomfortable with the $\pi(S)$ stuff, so I want to see if I can try to deduce it from that, just to get a better feel for it. We'll abbreviate $\pi(S) = g + t(k)$. Then our question is the number of pairs (t, k) that correspond to a numerical semigroup (necessarily of genus g).

First, let's describe some characteristics of S from $\pi(S)$. THIS IS WRONG: The multiplicity of S is k : I'll keep the following work because it's in the right spirit but it will be fixed in the section below. We go down to k and then right to $k + t$ and then down to $k + t + (g - k + 1) = g + t + 1$. That means

$$F(S) = g + t$$

which is a pretty nice formula, independent of k (which also makes sense if you think about the shape of the partition - the height of the parts doesn't change the number at the bottom, only adding more parts does).

And this again tells us that $1 \le t \le g - 1$. Choosing such a t, we get that $1 \le k \le g - t$. And for it to be a numerical semigroup with $\alpha = 3$, we'd need $2k \ge F(S) + 1 = g + t + 1$ and $k + t \le F(S) = g + t$, which gives the bounds

$$k + t + 1 \le F(S) + 1 \le 2k$$

Ok, cool, so the lower bound is achieved when $k = F(S) - t$ and the upper bound is achieved when $k = (F(S) + 1)/2$. If $F(S)$ is odd, then this means we have

$$(F(S) - t) - (F(S) + 1)/2 + 1 = \frac{1}{2}(F(S) - 3 - 2t)$$

plausible values for k. If $F(S)$ is even, then we have

$$(F(S) - t) - (F(S) + 2)/2 + 1 = \frac{1}{2}(F(S) - 4 - 2t)$$

plausible values.

Rewriting this in terms of g and t, we get

$$\begin{cases} \frac{1}{2}(g-t-4) & g \equiv t \ mod \ 2 \\ \frac{1}{2}(g-t-3) & g \not\equiv t \ mod \ 2 \end{cases}$$

To finish, we sum over t, but it'll be helpful to again take two cases based on the parity of g. If g is odd, then we get

$$\sum_{\substack{t=1 \\ odd}}^{g-2} \frac{1}{2}(g-t-4) + \sum_{\substack{t=2 \\ even}}^{g-1} \frac{1}{2}(g-t-3)$$

$$= \sum_{j=1}^{(g-1)/2} \frac{1}{2}(g-(2j-1)-4) + \sum_{j=1}^{(g-1)/2} \frac{1}{2}(g-(2j)-3)$$

$$= \frac{1}{2} \sum_{j=1}^{(g-1)/2} (g-2j-3+g-2j-3)$$

$$= \frac{1}{2} \sum_{j=1}^{(g-1)/2} (2g-4j-6)$$

$$= \sum_{j=1}^{(g-1)/2} (g-2j-3)$$

$$= \frac{g^2-8g+7}{4} = \frac{(g-7)(g-1)}{4}$$

And just for confirmation, since g is odd, we have $g \equiv 1, 3 \ mod \ 4$, so either $g-7 \equiv \ 0 \ mod \ 4$ or $g-1 \equiv \ 0 \ mod \ 4$.

I'm also not too surprised that this doesn't count the correct number for too small of g, since our assumptions needed some space. But for $g=9$, this gives 4, and we definitely have $n(9,3) > 4$, so what's going wrong?

Let's Start Again

Oh wow, I see what went wrong - very dumb, small mistake. If $\pi(S) = g + t(k)$, then we don't have $m(S) = k$, it's the *height* of the part that is k, which corresponds to a multiplicity of $m(S) = g-k+1$. I kept getting negatives! We can explicitly write

$$S \triangle (1+S) = \{0,\ 1,\ 1+g-k,\ 1+g-k+t,\ 1+g+t\}$$

For example, when $k=1$, we have

$$\{0,\ 1,\ g,\ g+t,\ g+t+1\}$$

So any t between 1 and $g-1$ will work. These correspond to the $L_{f,g}$ semigroups.

In general, we can have t between 1 and $g-k$, so we should have $g-k$ possible values of t for each k. Since $m \geq (F(S)+1)/2$, we get

$$g-k+1 \geq (F(S)+1)/2$$

$$k \leq g+1-(F(S)+1)/2$$
$$k \leq g+1-(g+t+1)/2$$
$$k \leq (g+1-t)/2$$
$$k \leq g/2$$

So in total, we should (maybe, hopefully), get that $n(g,3)$ is

$$\sum_{k=1}^{\lfloor \frac{g}{2} \rfloor}(g-k)$$

This would give

$$n(g,3) = \begin{cases} \frac{g(3g-2)}{8} & g \text{ even} \\ \frac{(g-3)(3g-1)}{8} & g \text{ odd} \end{cases}$$

Doesn't work! Looking at $g=4$ reveals the issue. We're counting $4+2$ and $4+2+2$ as valid, but the second corresponds to $\{1,2,5,6\}$, which isn't a valid semigroup. Out previous bound is $t \leq g-k$, which in this case would be $t \leq 2$, so that's clearly not actually true. But I think it's just off by 1? Or it could be off by $k-1$? I'm going pen and paper for a bit...

Let's Start Again Again

We're simply counting wrong (of course). If we fix a k, then it fixes a multiplicity $m = g-k+1$, and we need $2m > F(S) = g+t$, so $t < 2m-g$. So we get $2m-g-1$ choices for t. Taking the previous example of $g=4$ and $k=2$, we get $m=3$, so we have $2(3)-4-1=1$ choice for t. Great!

The bounds of $1 \leq k \leq g/2$ give the bounds $g/2+1 \leq m \leq g$. So **if g is even**, we have

$$\sum_{m=g/2+1}^{g} 2m-g-1 = \frac{g^2}{4}.$$

If g is odd, then we have

$$\sum_{m=(g+1)/2+1}^{g} 2m-g-1 = \frac{g^2-1}{4}.$$

Let's test: For $g=2$, we must have $m=2$, so $2m-g-1=1$.

For $g=3$, we can only have $m=3$, and we get $2m-g-1=2$ choices, which is good!

For $g = 4$, we can have $m = 3$, which gives $2m - g - 1 = 1$ choice (which is $\{1, 2, 4, 5\}$). We can also have $m = 4$, which has $2m - g - 1 = 3$ options ($\{1, 2, 3, 5\}, \{1, 2, 3, 6\}, \{1, 2, 3, 7\}$).

Jeez, that was tougher than it should have been! Let's go ahead and write the results for $\alpha = 1, 2, 3$ together:

Theorem 10 *For any $g \geq 2$, we have*

$$
\begin{aligned}
n(g, 1) &= 0 \\
n(g, 2) &= 1 \\
n(g, 3) &= \begin{cases} \frac{g^2}{4} & g \text{ even} \\ \frac{g^2-1}{4} & g \text{ odd} \end{cases}
\end{aligned}
$$

It's likely that $n(g, \alpha)$ is always a polynomial in g. But I can't quite guess yet the degree ($\alpha - 1$ seems too early to guess). I also want to see if there is some combinatorial way to see this, since $g^2/4$ is a surprisingly simple formula!

Our proof was to represent $\pi(S) = g + t(k)$ and count the number of pairs t, k that give a numerical semigroup (equiv. count possible pairs of multiplicity and Frobenius number). Then g^2 would come from taking $0 \leq t, k \leq g - 1$, and $g^2/4$ would come splitting them into four groups. We can actually picture the odd ones as completing a square with the even ones:

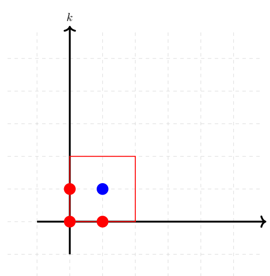


Figure 16: $g = 2$

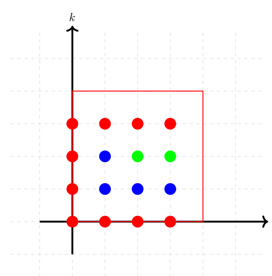


Figure 17: $g = 4$, $g = 3$

Since we want $\alpha = 3$, we can't have $t = 0$ or $k = 0$.

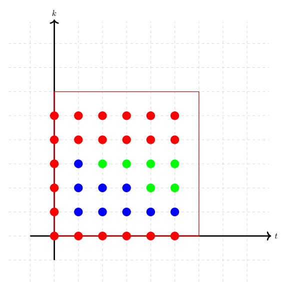


Figure 18: $g = 6$, $g = 5$

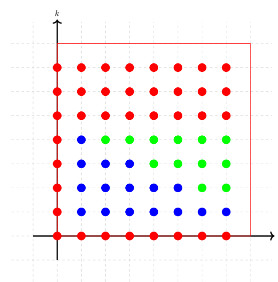


Figure 19: $g = 8$, $g = 7$

And to have the actual odd parts, we reflect the green dots to be in the right position. So really this is more just because the sum of consecutive odd numbers is a square, less so a division into four groups.

I want to go back to the bit around Wilf's conjecture and try to look further at the relation between $e(S)$ and $dim_\alpha(S)$. In some sense, they are dual to each other. Looking at our desired $\hat{0}$ and $\hat{1}$, we have

$$e(\{1, 2, \ldots, g\}) = g + 1 \qquad dim_\alpha(\{1, 2, \ldots, g\}) = 2$$

$$e(\{1, 3, \ldots, 2g - 1\}) = 2 \qquad dim_\alpha(\{1, 2, \ldots, g\}) = g + 1$$

To make this idea more exact, let's define

$$td(S) = e(S) + dim_\alpha(S)$$

to be the *total dimension* of S. In a moment, we'll enumerate $\mathcal{N}$ by total dimension. But for now, the point is that $td(S)$ seems to be fairly tightly bound by its genus. With the previous examples, we have

$$td(\mathcal{O}_g) = td(\{1, 3, \ldots, 2g - 1\} = g + 3$$

For each g, we'll list the range of the total dimension:

$$\begin{aligned} 4 &\leq td(\mathcal{N}_1) \leq 4 \\ 5 &\leq td(\mathcal{N}_2) \leq 5 \\ 5 &\leq td(\mathcal{N}_3) \leq 6 \\ 6 &\leq td(\mathcal{N}_4) \leq 7 \\ 5 &\leq td(\mathcal{N}_5) \leq 8 \\ 6 &\leq td(\mathcal{N}_6) \leq 9 \end{aligned}$$

And we'll note that for $g = 5$, the numerical semigroup $\{1, 2, 3, 6, 7\}$ gives the unusually low lower bound, which is interesting because it's also the numerical semigroup that first achieved a $D_S(-1)$ larger than 1.

Let's go ahead and look at numerical semigroups by total dimension. There are none until $td = 4$.

total dimension	numerical semigroups	count
4	$\{1\}$	1
5	$\{1,2\},\{1,3\},\{1,2,5\},\{1,2,3,5,7\}$	≥ 4
6	$\{1,2,3\},\{1,3,5\},\{1,2,3,6\},\{1,2,3,4,7,8\}$	≥ 9

It's hard to tell whether each class should even be finite! Let's look at some of the usual examples. How about $L_{f,g}$? Clearly it comes down to it's embedding dimension, since $\alpha = 3$ is fixed, but what is $e(L_{f,g})$ anyway? Well, as $g+1 \leq f \leq 2g-1$, we have

$$e(L_{f,g}) \geq n(L_{f,g}) = f - g$$

And really, any element in $L_{f,g}$ that is between $g+1, 2g-1$ would have to be a minimal generator, although we'd have to account for $f = g+1$ separately. Let's write it as a proposition.

Proposition 7 *The embedding dimension of the almost-ordinary semigroup $L_{f,g}$ is*

$$e(L_{f,g}) = \begin{cases} g & for\ f = g+1 \\ g-1 & for\ g+2 \leq f \leq 2g-1 \end{cases}$$

Proof 7 *Whatever f is, the elements $g, g+1, \ldots, f-1, f+1, \ldots, 2g-1$ all must be minimal generators, which is where the $g-1$ comes from. If $f > g+1$, then $2g+1 = g+(g+1)$, and $2g+2 = 2(g+1)$, and every further element will be reachable, so we only need those $g-1$ generators. If $f = g+1$, then we'll only have $2g$ and $g+(g+2) = 2g+2$, so we also need to include $2g+1$ as a minimal generator.*

So the take-away is that $td(L_{f,g}) = g+2$ or $g+3$, which means it's unbounded. That would be a fun way to show that the set

$$TD(k) = \{S\ numerical\ semigroup\ :\ td(S) = k\}$$

is finite for any fixed k: Show that if we fix α, the embedding dimension must increase as we get more numerical semigroups in $TD(k)$. Then show that if we fix the embedding dimension, α must increase.

With this in mind, can we describe $e(S)$ for a numerical semigroup S with $dim_\alpha(S) = 3$? Remember that we write $\pi(S) = g + t(k)$, and if we choose an $m = g - k + 1$, then we get $2m - g - 1$ choices for t. This let us count $n(g, 3)$, but let's dive deeper. We can write S as

$$\{1, 2, 3, \ldots, m-1, m+t, m+t+1, \ldots, g+t\}$$

and now we have the possibility that $2m = m + t + a \leq g + t$ for some a.

Let's separate the two cases. Let's suppose we have an m so that $2m > g+t$. Actually, let me rewrite this as $F(S) < 2m$, because this is exactly the condition on Zhao's result we mentioned way before! He proved the number of such numerical semigroups with genus g is the $g+1^{st}$ Fibonacci number. And we keep coming to this same condition in many different ways.

Anyway, in this case, we have all the t elements $m, \ldots, m+t-1$ are minimal generators, as well as the elements $g+t+1, \ldots, 2m-1$. Together, this gives us

$$e(S) \geq 2m - g - 1$$

Which actually gives a way to maybe produce infinitely many semigroups with bounded total dimension. If for growing g, we can choose an m so that $2m-g-1$ is small, which gives the possibility that $e(S)$ is small. But it is a lower bound.

The next case is where $2m < m+t$, which implies $m < t$. But $m+(m+1) = 2m+1$ also must be in S, so we'd need $2m+1 < m+t$ or $2m+1 > g+t$. If we're in the first case, we do this again, and reach the conclusion: If $2m < m+t$, then we will have some $s \in S$ so that $m+s = m+t$ (specifically, $s = t$). And this is because we can't "skip over" the second interval of gaps. This is actually pretty cool, because we can connect it to Zhao's result.

Proposition 8 *If $dim_\alpha(S) = 3$, then $F(S) < 2m$.*

Proof 8 *The above paragraph is a constructive, hands-on proof, but I'll give another one here. Suppose that $dim_\alpha(S) = 3$. Let's use our upper bound in Theorem 9:*

$$depth(S) \leq dim_\alpha(S) - 1$$

This means

$$\frac{F(S)+1}{m(S)} \leq 2$$

and so

$$F(S) \leq 2m(S) - 1$$

Corollary 4 *For all $g \geq 3$, the number of numerical semigroups S of genus g that have $F(S) < 2m$ is at least $\frac{g^2-1}{4}$*

Which is of course weaker than Zhao's result since $(g^2-1)/4 < Fib(g+1)$, but we can view it as counting the contribution of α-dimension 3 semigroups. We should definitely see if we can count the contributions from other α and get the full Fibonacci number.

And I've got to point out something cool when we take $dim_\alpha(S) = 4$. Then our bound ends up being

$$F(S) < 3m$$

In the paper we keep referencing by Nathan Kaplan, he begins exactly with those numerical semigroups with $F(S) < 3m(S)$. He discusses Zhao's result that gives the number of numerical semigroups with $F(S) < 3m(S)$ as a sum of Fibonacci numbers, which gives the asymptotic growth.

So analyzing when $dim_\alpha(S) = 4$ will give a different (probably worse) lower bound on the number of numerical semigroups with $F(S) < 3m(S)$. And setting $\alpha = 5$ would give a lower bound on the number of semigroups with $F(S) <$

$4m(S)$, and so on. Which means as we increase α, we should be getting a number that's closer to the actual count of numerical semigroups! Although if $n(g, \alpha)$ truly is a polynomial for all α, then

$$\lim_{g \to \infty} \frac{n(g, \alpha)}{Fib(g+1)} = 0$$

for all α. But

$$N(g) = \sum_{\alpha} n(g, \alpha)$$

and we only have finitely many terms. Zhao's work tells us

$$\lim_{g \to \infty} \frac{N(g)}{Fib(g+1)} \neq 0$$

which means we must have some α for which

$$n(g, \alpha) = O(Fib(g+1))$$

which can't be a polynomial.

The above in red is definitely false, with a simple counter-example. If $f(x) = 1$ and $g(x) = x$, then $f(x) = o(g(x))$, but

$$\sum_{a=1}^{x} f(x) = \sum_{a=1}^{x} o(g(x)) = o(xg(x))$$

and we don't actually have $\sum f(x) = o(g(x))$.

So this actually means that we might have $n(g, \alpha) = o(Fib(g))$ for all α. This would imply

$$N(g) = o(gFib(g))$$

which follows from Zhao's Theorem, so we don't actually run into an issue.

But I guess this doesn't really account for the issue we ran into before anyway. And it's because the "finite sum" really contains arbitrarilly many values of α eventually. So really we get a weaker result that $n(g, \alpha)$ cannot be a polynomial in g with bounded degree over α. If so, then letting

$$M = \max_{\alpha}(deg(p_\alpha)),$$

we have

$$n(g, \alpha) = o(g^{M+1})$$

for all α, which by summing, would give

$$N(g) = o(g^{M+2})$$

which we know is false from Zhao.

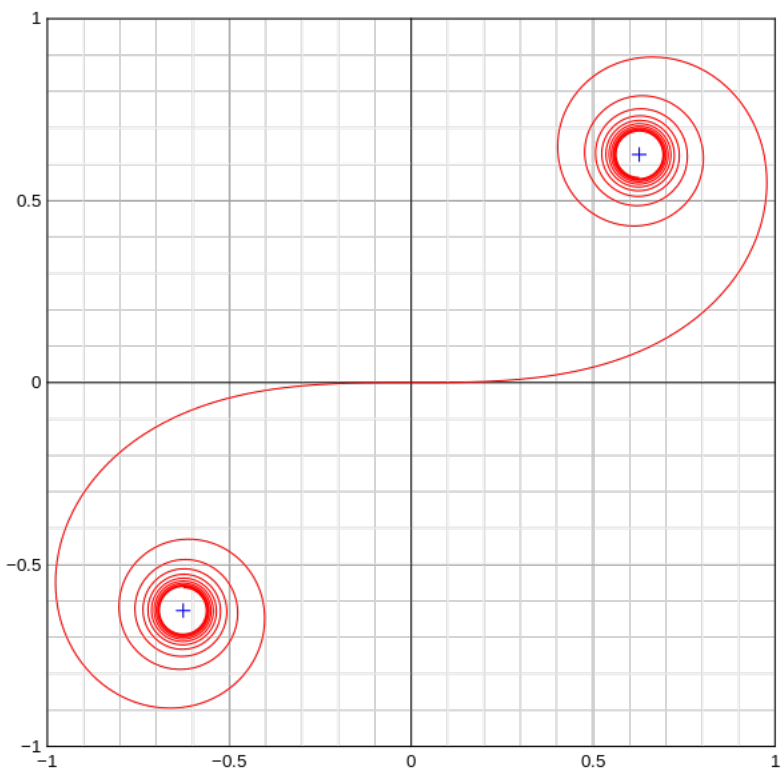
If (NOT TRUE) for example $n(g, \alpha) = g^\alpha$, then we would have $N(g) = \sum_{\alpha=1}^{g+1} g^\alpha = g\left(\frac{g^{g+1}-1}{g-1}\right)$, which is an exponential function that dominates $Fib(g)$.

So there is some in-between...a polynomial in g whose growth of degree in α is slow enough to get $N(g) = O(Fib(g))$.

I mean, this is really true for any statistic we're talking about - If we know $N(g)$ (asymptotically), then what restrictions do we have to have on $n(g, st)$ the number of semigroups of genus g with fixed statistic? I'm going to go look through some papers/books and see if I can find a nice exposition of this kind of thing.

I'm going to look through a short course on asymptotics for an REU at the University of Illinois with AJ Hildebrand. I remember getting a similar 50-page packet at the start of my REU at Michigan State with Bruce Sagan. It was packed with information but we got through it together in about a week and then went wild trying to apply it.

His initial examples of asymptotics is a great refresher and I'm learning new stuff already! For example, Fresnel Integrals and their resulting graphs being the beautiful Cornu Spiral (a.k.a Euler Spiral).



And on page 16, he gives some properties of O-estimates that follow from the definition, which I'll summarize here:

1. $O(Cf(x)) = O(f(x))$

2. $O(f_1(x)) + O(f_2(x)) = O(|f_1(x)| + |f_2(x)|)$
3. $O(f_1(x)) - O(f_2(x)) = O(|f_1(x)| + |f_2(x)|)$ (so O-estimates never cancel out)
4. And here's some confirmation for the O-sums we are working on. We have that
$$O\left(\sum_n f_n(x)\right) = \sum_n O(f_n(x)))$$
BUT ONLY IF the O-estimate is uniformly bounded in n, meaning there is some constant c independent of n so that
$$|f_n(x)| \leq c|g_n(x)|$$
for all n.
5. Finally, he mentions distribution, which says
$$h(x)(O(f(x)) + O(g(x))) = O(|h(x)|(|f(x)| + |g(x)|))$$

And then there are many wonderful examples and tricks - essentially it seems like an incredibly helpful pre-REU packet!

13 Exploring Duality

I've been using duality very loosely, but there is of course a more precise mathematical notion. In the most basic sense, duality is an involution. More often than not, we want it to be an involution that translates properties between two objects.

For example, taking the complement of a set is a type of duality. We have $(S^c)^c = S$, and properties about maps between sets are transferred from a set to its complement. And the transfer is *contravariant*, since an inclusion $S \hookrightarrow T$ gives an inclusion $T^c \hookrightarrow S^c$. Another place we've seen this is in discussing the spectrum of a ring, since transferring the property of "prime ideal" is contravariant.

Another important instance of duality is in linear algebra. Given a vector space V over a field K, we can form the *dual vector space* V^* as the space of *linear functionals* $\varphi : V \to K$. This is also a contravariant duality, and it's not at all clear that $(V^*)^*$ should be the same as V. And it turns out they are isomorphic, but not *naturally isomorphic*. This means that the isomorphism depends on a choice (in this case, of basis), but more generally is a statement about categorical equivalence. And although transposing a matrix is a simple operation to learn, it actually transferring to the dual space!

There is also duality in Galois Theory, but it is (at least partially, I think) encompassed by duality of posets, which essentially corresponds to flipping all the arrows. Although that sounds simple, there are so many important concepts that duality allows us to view in a more unified perspective:

- Minimum elements and maximum elements of a poset.
- Upper bounds and lower bounds.
- Ideals (downward-closed sets) and filters (upwards-closed sets)
- Maximal Independent Sets and Clique Number ($Ind(G) = Clique(G^c)$)
- Adjoint Operators

Now the involution doesn't have to be a bijection. We could have fixed points, and those are often the points of interest! There are dozens of fixed-point theorems to count fixed points of various types of maps between various types of objects.

As an example of how useful this can be, Don Zagier (who did much of the work my first number theory research was based on) wrote a one-sentence proof of Fermat's theorem on primes as the sum of squares. Here it is:

A One-Sentence Proof That Every Prime $p \equiv 1 \pmod 4$ Is a Sum of Two Squares

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Department of Mathematics, University of Maryland, College Park, MD 20742

The involution on the finite set $S = \{(x, y, z) \in \mathbb{N}^3 : x^2 + 4yz = p\}$ defined by

$$(x, y, z) \mapsto \begin{cases} (x + 2z,\ z,\ y - x - z) & \text{if } x < y - z \\ (2y - x,\ y,\ x - y + z) & \text{if } y - z < x < 2y \\ (x - 2y,\ x - y + z,\ y) & \text{if } x > 2y \end{cases}$$

has exactly one fixed point, so $|S|$ is odd and the involution defined by $(x, y, z) \mapsto (x, z, y)$ also has a fixed point. □

Lovely!

So what's all this about? Well, if we really want to cement the idea that $e(S)$ and $dim_\alpha(S)$ are dual to eachother, we should find some involution that swaps them. The notion comes from the idea that **ideals** in rings and **adeals** in rings are "dual" in the sense that they're both "sponges" ($SI \subseteq I$ and $S + A \subset A$). But this isn't quite the normal sense of duality. Do we have some map between them? We have a duality in the same sense in the following theorems. We'll write $ad(S)$ for the set of adeals of a semiring and $id(S)$ for the set of ideals. Then the two theorems are

$$id(S) \textit{ is trivial} \iff S \textit{ is a field.}$$

$$ad(S) \textit{ is trivial} \iff S \textit{ is a ring.}$$

And assume we're always talking about commutative semirings.

The statement above make me want to try to abstract things a bit, but warning, I'm not terribly confident how this will go! First, recall a binary operation on a set S is a function $f : S \times S \to S$ that tells us how to combine two objects of S.

Definition 7 An *f-absorbing subset* of S is a non-empty subset A in which

$$f(S, A) \subseteq A$$

where $f(S, A) = \{f(s, a) \ : \ s \in S, a \in A\}$.

As the definition was directly inspired, we have two examples: *ideals are product-absorbing subsets of a semiring* and *adeals are sum-absorbing subsets of a semiring.* But we have more!

Suppose we took our set $S = 2^X$ to be the power set of some X with n elements, and took our operation to be the union. Then A being an absorbing set would mean

$$s \cup a \in A$$

for all $s \in S$ and $a \in A$. An example would be A_m, the set of subsets of X with size at least m, because unioning can't decrease size. And for any subset $T \subset X$, we could form a *principal absorbing set* $A(T)$ containing all sets with T as a subset.

We've mentioned filters a few times so let's define it. A *filter* F of a partially-ordered set $(P, \leq)$ is a subset satisfying:

1. F is non-empty.

2. F is closed under finite intersections: for all $x, y \in F$, there exists $z \in F$ so that $z \leq x$ and $z \leq y$.

3. F is closed under union by P: for all $p \in P$ and $f \in F$, if $f \leq p$ then $p \in F$.

The definition is very topological. The second condition is what separates filters from what we're calling *union-absorbing sets.* For example, the principal absorbing set $A(T)$ is also a principal filter. But the absorbing sets A_m are not filters because they are not closed under intersection. So every filter is an absorbing set, but not vice versa. The duality between filters and ideals could perhaps be extended?

Another operation on $S = 2^X$ is intersection, so what's an *intersection-absorbing set*? Well it'd have to be some $A \subseteq S$ so that

$$s \cap a \in A$$

for all $s \in S$ and $a \in A$. We can form them in the dual way to before (via set complement and union/intersection).

What about the symmetric difference? What's a $\triangle$-absorbing set? Well, I claim that it's trivial! The only $\triangle$-absorbing set is S itself. Let's call such a set A and see what it must contain. First, it contains some set T, so it must contain $T \triangle T = \{\}$. This is the identity element for the operation, like 1 for multiplication and 0 for addition and $\{\}$ for unions and X for intersections. Meaning, A must contain $s \triangle \{\} = s$ for all $s \in S$, so $A = S$.

The previous example reveals the "general form" of the initial two theorems we discussed for motivation. Suppose that S has an identity element 1 under f, meaning $f(s,1) = s$ for all $s \in S$. Let $ab_f(S)$ be the collection of f-absorbing subsets of S. Then

$$ab_f(S) = \{S\} \iff \textit{Every element of } S \textit{ is invertible under } f$$

which comes from the fact that every non-invertible element $a \in S$ generates an f-absorbing set $\{f(s,a) \ : \ s \in S\}$, which is equal to S if and only if it contains 1, which means there is some $a' \in S$ so that $f(a',a) = 1$, which means a is invertible.

I realized something I should make clear: I am assuming the binary operation is symmetric.

Now if we have multiple binary operations, we can look at how they interact. In this function notation, distribution $a * (b + c) = (a * b) + (a * c)$ looks like

$$f_1(f_2(b,c),a) = f_2(f_1(b,a), f_1(c,a))$$

which definitely make it clear why we use the clearer notation - but this will be helpful to not have to write strange indexed binary operations as $*_1, *_2$ or something.

Let's suppose we have a ring R with addition and multiplication and $id(R)$ is its set of ideals. Then a standard result is

$$id(R) = \{(0), R\} \iff R \textit{ is a field}$$

And in this case, we'd consider a field to be a set closed under one operation (addition) and *almost* closed under another (multiplication), with the exception of the identity of the first operation (0).

Let's get more grounded for a second and talk about the adeals of $\mathbb{N}$, since I realize we haven't done that yet and it's the most basic case. By the definition of an adeal, if we contained $n \in \mathbb{N}$, then we must contain all integers above n. So adeals "look" similar, but I'll write $A(T)$ for the adeal generated by a set T and $I(T)$ for the ideal generated by T, to distinguish the two easier.

$$id(\mathbb{N}) = \{I(n) \ : \ n \in \mathbb{N}\}$$
$$ad(\mathbb{N}) = \{A(n) \ : \ n \in \mathbb{N}\}$$

where $I(n) = \{nk \ : \ k \in \mathbb{N}\} = n\mathbb{N}$ and $A(n) = \{n+k \ : \ k \in \mathbb{N}\} = \{n, n+1, \dots\}$.

I just found the final section of the wikipedia page on ideals talks about a generalization just like what we want! It deals with a *monoid object*, which is essentially (don't hate me) a category with a notion of multiplication of objects.

It's actually this point of view that brought me to what seems like a beautiful connection that I still don't quite understand, back when I was writing my NSF GRFP statement about my research with Cassie Williams. Perhaps, instead of trying to re-invent the wheel, let's dive into some of the category theory we know helps here. This will also give me an opportunity to refresh myself on the previously mentioned connection.

You can read my personal and research statement for the NSF GRFP, as well as my official reviews, on my old blog. Even I don't think I could summarize it here right now - it's very dense! But in time, we'll get there. Some keywords to explore:

- Divisors of algebraic curves and configurations in the critical group.
- Riemann-Roch on Graphs
- The Picard/Jacobian group and the Critical group.
- Similarly, the ideal class group and the class number/relative class number.
- Principally Polarizable Ordinary Abelian Varieties (and abelian varieties in general).
- Conjugacy Classes in the General Symplectic Group.
- People: Matt Baker, Serguei Norine, Ernst-Ulrich Gekeler, Jeff Achter, Gregg Musiker, Andre Neron (who I'm fairly certain explicitly formed this type of connection, but I can't find the paper currently!).

A wonderful reference to get a feel for the more-difficult stuff is always Keith Conrad's Expository Papers. This page helped get me understand many many of the concepts important to Number Theory and onward. I think I want to start here with Factoring in Quadratic Fields, as this is the "next step" after $\mathbb{Z}$.

For ideals, we can define sums and intersections, which are also ideals:

$$I + J = \{i + j \ : \ i \in I, j \in J\}$$

$$I \cap J = \{x \ : \ x \in I, x \in J\}$$

If for the product, we simply take ij for all $i \in I$ and $j \in J$, we won't get additive closure, so we actually have to take all finite linear combinations. In other words, IJ is defined to be the smallest ideal containing $\{ij \ : \ i \in i, j \in J\}$.

If we have explicit generators, then we can take the ideal generated by all pairs of generators multiplied together. For example, $I(2, x)$ and $I(3, x)$ are two ideals in $\mathbb{Z}[x]$, and

$$I(2, x)I(3, x) = I(6, 2x, 3x, x^2) = I(6, x),$$

since $3x - 2x = x$.

TBC

14 More Numerical Semigroups

I came across another thing I looked at very briefly in the past, but it looks interesting, so let's bring it up here!

For a principal ideal $(n) \in \mathbb{N}$, its cosets $k + (n) = \{An + k \ : \ A \geq 0\}$ are all disjoint. But any numerical semigroup will have "cosets" that actually overlap a bit. For example, the cosets of $S = \{1\}^c$ are

$$S = \{0, 2, 3, 4, \dots\}$$

$$1 + S = \{1, 3, 4, 5, \dots\}$$

$$2 + S = \{2, 4, 5, 6, \dots\}$$

and so on. Notice that once we get to 2, we have $2 + S \subset S$. And this is always true - since S is an ideal, $m(S) + S \subset S$. To study the overlap, we can use the symmetric difference again! The overlap in cosets here is encoded in

$$S \triangle (1 + S) = \{0, 1, 2\}$$

(and this is actually where they don't overlap).

We've worked a lot with various types of gaps, often depending on what happens to them when an element of S or a gap is added to them. So we've already been thinking of these near-cosets. For example, a *fragile gap* a was one so that $s + a \in S$ for all $s \in S \cup \{a\}$.

To try to study all of these shifts at once (no promises), we'll define $T(S)$ to be the symmetric difference of all shifts of S up to $m(S) - 1$, i.e.

$$T(S) = S \triangle (1 + S) \triangle (2 + S) \triangle \dots \triangle (m(S) - 1 + S)$$

The reasoning for this is the following theorem:

$$T(S) \neq \mathbb{N} \iff S \textit{ numerical semigroup}$$

To see this, if S is simply a principal ideal (s), then we already discussed that the cosets $k + (s)$ are all disjoint, so their symmetric difference is

$$\bigcup_{k=0}^{m-1} (k + (s)) = \mathbb{N}$$

For the numerical semigroup

$$S = I(2, 2g + 1) = \{0, 2, 4, \dots, 2g - 2, 2g, 2g + 1, \dots\},$$

we have a single shift

$$1 + S = \{1, 3, 5, \dots, 2g - 1, 2g + 1, 2g + 2, \dots\}$$

so

$$T(S) = \{0, 1, 2, \dots, 2g - 1, 2g\}.$$

Let's do another example of

$$S = \{0, 6, 7, 8, \dots\}$$

$$1 + S = \{1, 7, 8, 9, \dots\}$$

$$2 + S = \{2, 8, 9, 10, \dots\}$$

$$3 + S = \{3, 9, 10, 11, \dots\}$$

$$4 + S = \{4, 10, 11, 12, \dots\}$$

$$5+S=\{5,11,12,13,\dots\}$$

$$T(S)=\{0,1,2,3,4,5,6,\ 8,10\}$$

Let's be more explicit about what this repeated symmetric difference is: We take the elements that appear in an odd number of cosets. As such, we know that each non-negative integer between 0 and $m-1$ will be in $T(S)$, as they'll appear in each coset exactly once. And we actually know that $m(S)$ will always be in $T(S)$ as well, since it can only appear in S and none of its shifts. We also know that every integer larger than $F(S)+m(S)-1$ will appear in all of the shifts, so $T(S)$ will contain $[F(S)+m(S),\infty)$ as long as $m(S)$ is odd. These forced elements are colored blue above. Note the missing 7 and 9.

If we have a general ordinary numerical semigroup $S=\mathcal{O}_g=\{0,g+1,g+2,\dots\}$ for $g\geq 2$, then we have g shifts of the form

$$k+S=\{k,k+g+1,k+g+2,\dots\}$$

for $0\leq k\leq g-1$. To figure out which $x\in T(S)$, we have $[0,g+1]$ by default, so we can assume $x\geq g+1$. Then $x\in k+S$ for all $k\leq x-g+1$. So for $g+1\leq x\leq 2g$, we have

$$x\in T(S)\iff x-g+2\ is\ odd\iff x\not\equiv g\ mod\ 2$$

For the previous example, $g=5$ is odd, so we have the even numbers in $[7,10]$, which is where the $8,10$ comes from.

In general,

$$T(\mathcal{O}_g)=\begin{cases}[0,g+1]\cup\{g+3,g+5,\dots,2g\} & g\ odd\\ [0,g+1]\cup\{g+3,g+5,\dots,2g-1\}\cup[2g+1,\infty) & g\ even\end{cases}$$

Since we did the odd case, let's do an even example with $g=4$.

$$S=\{0,5,6,7,\dots\}$$

$$1+S=\{1,6,7,8,\dots\}$$

$$2+S=\{2,7,8,9,\dots\}$$

$$3+S=\{3,8,9,10,\dots\}$$

$$4+S=\{4,9,10,11,\dots\}$$

$$T(S)=\{0,1,2,3,4,5,\ 7,9,\dots\}$$

Back to the general situation, the values that aren't forced to be included are $[m(S)+1,F(S)+m(S)-1]$, so let's start with $m(S)+1$. Well, it'll appear in $1+S$, and can't appear anywhere else except maybe S. Therefore,

$$m+1\in T(S)\iff m+1\in S\triangle(1+S)\iff m+1\notin S$$

On the flip-side, the value $F(S)+m(S)-1$ will be in all shifts except for the last, $m(S)-1+S$, so

$$F(S)+m(S)-1 \in T(S) \iff m(S) \textit{ is even}$$

And I guess a way to say the final infinite bit $[F(S)+m(S), \infty)$ is contained in S when $m(S)$ is odd would be

$$F(S)+m(S) \in T(S) \iff m(S) \textit{ is odd}$$

So we had a forced initial sequence, an undetermined center, and an end sequence whose existence depends on the parity of $m(S)$. Then the undetermined center is based off the shape of the numerical semigroup. (As such, it's probably worthwhile to see how $T(S)$ changes under shifts/adding blocks)

Let's take an element $m(S)+2 \leq N \leq F(S)+m(S)-2$ and see how it could be in $T(S)$. There'd have to be an odd number t of shifts $k_1, k_2, \dots, k_t$ so that $N \in k_i + S$. Which means there are distinct $s_1, \dots, s_t \in S$ so that

$$k_i + s_i = N$$

for all i.

A statement similar to this is Schur's theorem on the number of representations of a number x as the sum of relatively prime integers. I covered it in the last document, but this proof is found on page 98 of Generatingfunctionology by Herbert Wilf. But that's asymptotic and we need something more exact.

Let's try to turn to generating functions, inspired by Wilf. We'd want the generating function for the shifts:

$$K(S,x) = \sum_{i=0}^{m(S)-1} x^i = \frac{x^{m(S)}+1}{x-1}$$

And we'd want the generating function for S:

$$L(S,x) = \sum_{s \in S} x^s$$

Then we have

$$N \in T(S) \iff \textit{coefficient of } x^N \textit{ in } K(S,x)L(S,x) \textit{ is odd}$$

To see this more clearly, let's explicitly multiply it out:

$$K(S,x)L(S,x) = \left(\sum_{i=0}^{m(S)-1} x^i\right)\left(\sum_{s \in S} x^s\right) = \sum_{i=0}^{m(S)-1} \sum_{s \in S} x^{i+s}$$

$$= \sum_{N \geq 0} c_N x^N$$

where c_N is the number of pairs $(i,s) \in [0,m-1] \times S$ with $i+s=N$. Thinking of i as a shift, we just get exactly what we were talking about before - we have N in $T(S)$ if and only if c_N is odd.

This is pretty fun because it means we can get a generating function for $T(S)$ by looking over $\mathbb{F}_2$:

$$K(S,x)L(S,x) = \sum_{s \in T(S)} x^s = \frac{x^{m(S)}-1}{x-1} L(S,x)$$

This feels like a much simpler way to generate $T(S)$ to be honest, since doing successive symmetric differences is annoying. And this might actually give a good way to describe what happens to $T(S)$ under shifts and other operations.

For example, recall that we defined the shift of a numerical semigroup to be

$$\phi(S) = (1+S) \cup \{0\} - \{1\}$$

so that if S is genus g, then $\phi(S)$ had the chance to be a numerical semigroup of genus $g+1$. I'll mention again here the conjecture that the number of shifted and unshifted numerical semigroups is always equal, or off by 1, for all genus.

Then

$$L(\phi(S),x) = \sum_{s \in \phi(S)} x^s = 1 - x + \sum_{s \in S} x^{s+1} = 1 - x + xL(S,x)$$

where we do note that $L(S,x)$ being infinite series means we need $|x| < 1$ if we ever want to evaluate the sum, and we can employ an Egorychev Method type argument to look at the residue at 1 or ∞.

This looks similar to the formula on page 89 for $D_{\phi(S)}(x)$, since we did pretty much the same thing. Let's iterate the shifts a few times

$$L(\phi^2(S),x) = 1 - x + x(1 - x + xL(S,x)) = 1 - x^2 + x^2 L(S,x)$$

and

$$L(\phi^3(S),x) = 1 - x + x(1 - x^2 + x^2 L(S,x)) = 1 - x^3 + x^3 L(S,x)$$

and in general, we have

$$L(\phi^k(S),x) = 1 - x^k + x^k L(S,x)$$

So let's look at a few evaluations. If we choose $x = \zeta_k$ some k^{th} root of unity, then

$$L(\phi^k(S),\zeta_k) = L(S,\zeta_k)$$

The fact that these have magnitude 1 though means that these might just both be infinite.

Having mentioned the polynomial $D_S(x)$, I kind of want to go back and have a look at iterating the shift operator. Perhaps that can help answer the conjecture that

$$N(g-1) \le \sum_{S \in \overline{\mathcal{N}}_g} D_S(-1) \le N(g)$$

To recall,

$$D_S(x) = \frac{1}{2}\left(1 + \sum_{n \in S \triangle (1+S)} x^n\right)$$

encoded the corners of the partition $\pi(S)$. The scaling factors come from forcing $D_S(1)$ to be the number of inner corners. It's unclear what $D_S(-1)$ counts, but we did find the recursions:

$$D_{\phi(S)}(x) = xD_S(x) - \frac{(x+2)(x-1)}{2}$$

$$D_{S^\ell}(x) = D_S(x) + \frac{x^{s-1} - x^{s+1}}{2}$$

$$D_{S^r}(x) = D_S(x) + \frac{x^{s+1} - x^{s-1}}{2}$$

$$D_{S^d}(x) = D_S(x) + \frac{x^{s-1} + x^{s+1}}{2}$$

where S_ℓ, S^r, and S^d are the sets after adding a left/right/deep corner to S.

So iterating the shift operation would give

$$D_{\phi^2(S)}(x) = x\left(xD_S(x) - \frac{(x+2)(x-1)}{2}\right) - \frac{(x+2)(x-1)}{2}$$

$$= x^2 D_S(x) - x\frac{(x+2)(x-1)}{2} - \frac{(x+2)(x-1)}{2}$$

$$= x^2 D_S(x) - (x+1)\frac{(x+2)(x-1)}{2}$$

And for the next shift,

$$D_{\phi^3(S)}(x) = x\left(x^2 D_S(x) - (x+1)\frac{(x+2)(x-1)}{2}\right) - \frac{(x+2)(x-1)}{2}$$

$$= x^3 D_S(x) - (x^2 + x + 1)\frac{(x+2)(x-1)}{2}$$

And the general pattern becomes apparent,

$$D_{\phi^k(S)}(x) = x^k D_S(x) - (1 + x + \cdots + x^{k-1})\frac{(x+2)(x-1)}{2}$$

$$= x^k D_S(x) - \frac{(x^k - 1)(x+2)}{2}$$

Plugging in a k^{th} root of unity ζ_k this time gives

$$D_{\phi^k(S)}(\zeta_k) = D_S(\zeta_k)$$

We've seen both functions are equal if you plug in the same root of unity as the shift, and essentially what I'd like is if plugging in a different root of unity does not give equality. This would give a way to test whether $S' \in \overline{\mathcal{N}}_{g+k}$ if a shift of some $S \in \mathcal{N}_g$.

Here are both recursions with the shifts:

$$L(\phi^k(S), x) = 1 - x^k + x^k L(S, x)$$

$$D_{\phi^k(S)}(x) = x^k D_S(x) - \frac{(x^k - 1)(x + 2)}{2}$$

15 Building Semigroups

Ok, I want to try to organize the thoughts related to $D_S(-1)$ being bounded between $N(g-1)$ and $N(g)$, as this did give me a more general proof idea that I realized I hadn't explicitly thought about before - To show $f(n) < f(n+1)$, we could find $g(n)$ so that $f(n) < g(n)$ and $g(n) < f(n+1)$.

With the main object of interest being $D_S(-1)$, let's rewrite the change in $D_S(x)$ when we try to build a semigroup via its partition, by adding blocks.

$$D_{\phi^k(S)}(x) = x^k D_S(x) - \frac{(x^k - 1)(x + 2)}{2}$$
$$D_{S^\ell}(x) = D_S(x) + \frac{x^{s-1} - x^{s+1}}{2}$$
$$D_{S^r}(x) = D_S(x) + \frac{x^{s+1} - x^{s-1}}{2}$$
$$D_{S^d}(x) = D_S(x) + \frac{x^{s-1} + x^{s+1}}{2}$$

And let's rewrite the idea behind the left-deep,right-deep, and deep inner corners.

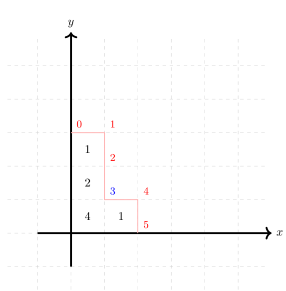


Figure 20: Left-deep corner

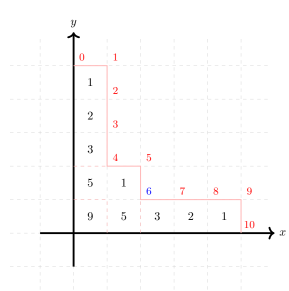


Figure 21: Right-deep corner

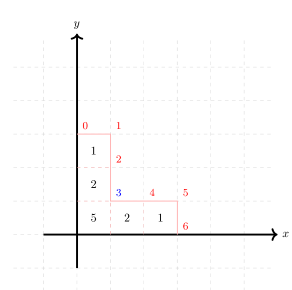


Figure 22: Deep corner

The effect these have when plugging in -1 is that the shifting one alternates back and forth (as we already talked about), the left-deep corner and right-deep corner does not change anything (since $s+1$ and $s-1$ are always the same parity), but adding to a deep corner will change $D_S(-1)$ directly by 1 or -1, depending on the parity of the element added.

So starting at $\{1\}$, a single block, we can build up our partition by adding blocks of various types, increasing $D_S(-1)$ by 1 when we add to a odd deep corner and decreasing by 1 when we add to an even deep corner. We can think of these as strings with alphabet ℓ, r, d, ϕ for the chain of operations it takes to get to S. As we are genus g, there will be exactly $g-1$ copies of ϕ.

The question then is how the other three letters affect the final value of $D_S(-1)$. Of course, we've discussed that only d's will affect increases and decreases. So could we have two strings corresponding to the same numerical semigroup with a different number of d's? How many strings do we have that correspond to a single semigroup?

That number of strings is interesting enough (i.e. I think we'll get some cool combinatorics from it) that I want to go ahead and give it a name: *I'll write $\mathfrak{t}(S)$ for the number of such strings that build S from the single block $\{1\}$ semigroup.*

So $D_S(-1)$ is not the same as $\alpha_d(S)$, the number of deep corners of S. It's kind of a count of the number of deep corners that have to be filled, and whether they're even or odd, to get to S. It's surprising that this wouldn't depend on the path, but clearly $D_S(-1)$ is a single, unchanging value.

Ok, let's check out the alternating $\mathcal{O}_g$. We begin with $g=1$, the single block, with $D_S(-1)=1$. And this numerical semigroup has the unique string

$$\underbrace{\phi\phi\ldots\phi}_{g-1},$$

so $\mathfrak{t}(\mathcal{O}_g)=1$ for all g. From this point of view, the L-semigroups are the next simplest. The "almost-ordinary" numerical semigroup $L_{f,g}$ has length $f-g$ and height g, so we'll have an initial string ϕ^t of building blocks up, and then we'll have a single d. After that, we need $f-g-1$ copies of r and $g-1-t$ copies of ϕ, and we can choose the order with one restriction: The length always must be less than or equal to the height.

In terms of the strings, this means that we can fix any point in the string and it will always have less r's than ϕ's before that point. But before trying to count such strings, let's see what happens with $D_S(-1)$. With just the initial string, we'll be at $\mathcal{O}_{t+1}$, which has $D_S(-1)=1$ if t is even and 0 if t is odd.

And we'll recall that left-deep and right-deep corners don't change $D_S(-1)$, while

$$D_{\phi(S)}(-1) = (-1)D_S(-1)+1$$
$$D_{S_s^d}(-1) = D_S(-1)+(-1)^{s-1}$$

where S_s^d is adding to a deep corner with label s.

So we started with the obvious observation that

$$D_S(-1) = (1/2)(1 + EC(S) - OC(S))$$

where EC and OC is the number of even and odd labeled corners respectively. And now we've distilled it down further: The parity business comes solely from adding deep corners - no other building block cares about the label on that corner.

I need to make a list to keep my head straight:

1. Let's define the *embedded deep corners* of S to be all s so that S_s^d appears in some string.

2. Do all strings contain the same amount of d's? I asked it before, but it affects the definition we just made. Is it well defined if we change it to "in a fixed string"?

3. $\phi(S)$ is genus $g+1$ and S_s^d and S will both be genus g. If we add the above equations, we get

$$D_{\phi(S)}(-1) + D_{S_s^d}(-1) = 1 + (-1)^{s-1}$$

4. If we shift a set we added a deep corner to, we get

$$\begin{aligned} D_{\phi(S_s^d)}(-1) &= (-1)D_{S_s^d}(-1) + 1 = (-1)(D_S(-1) + (-1)^{s-1}) + 1 \\ &= (-1)D_S(-1) + (-1)^s + 1 \end{aligned}$$

5. If we add a deep corner to a shifted set, we get

$$D_{\phi(S)_{s'}^d} = D_{\phi(S)}(-1) + (-1)^{s'-1}$$

$$= (-1)D_S(-1) + 1 + (-1)^{s'-1}$$

   which is the same as the last one since if s is a deep corner of S, then $s+1$ is a deep corner of $\phi(S)$, so $s' - 1 = s$. All of this is to say that shifting and adding deep corners commute with each other (which is what we'd expect from the partition perspective)

And how about the staircase semigroup $\{1, 3, 5, \ldots, 2g-1\}$? It always has $D_S(-1) = 1$, which is easy to see by the actual polynomial, but how does it relate to embedded deep corners? Well the first is $\{1, 3\}$, which must have the string ϕd, since

$$\{1\} \xrightarrow{\phi} \{1, 2\} \xrightarrow{d} \{1, 3\},$$

and the deep corner added is 3, meaning

$$D_{\{1,3\}}(-1) = (-1)D_{\{1,2\}}(-1) + (-1)^2 = 0 + 1 = 1$$

It feels like we could just have all $g-1$ ϕ's at the beginning, but I want to not assume that until we can show that's possible - i.e. that there isn't some semigroup of genus g whose string *has to* go through a non-ordinary semigroup of genus $k < g$.

So what about $\{1,3,5\}$? We could go

$$\{1\} \xrightarrow{\phi} \{1,2\} \xrightarrow{\phi} \{1,2,3\} \xrightarrow{d_4} \{1,2,4\} \xrightarrow{r_5} \{1,2,5\} \xrightarrow{d_3} \{1,3,5\}$$

I'm also labeling the corner values when we add corners, just for added clarity. The $D_S(-1)$ values go

$$1 \xrightarrow{\phi} 0 \xrightarrow{\phi} 1 \xrightarrow{d_4} 0 \xrightarrow{r_5} 0 \xrightarrow{d_3} 1$$

We could have also done

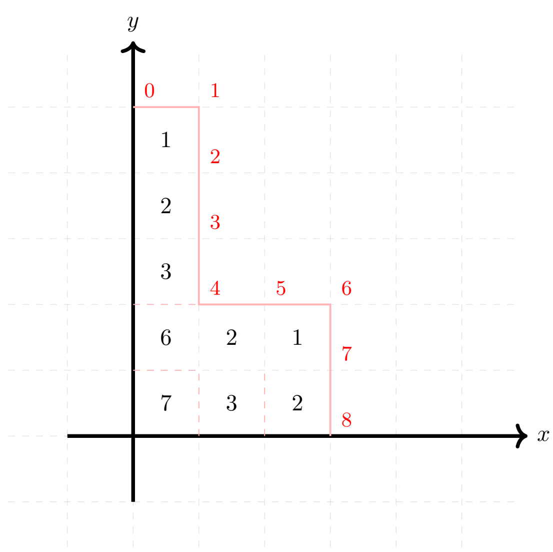
$$\{1\} \xrightarrow{\phi} \{1,2\} \xrightarrow{d_3} \{1,3\} \xrightarrow{\phi} \{1,2,4\} \xrightarrow{r_5} \{1,2,5\} \xrightarrow{d_3} \{1,3,5\}$$

which would give the sequence of $D_S(-1)$ as

$$1 \xrightarrow{\phi} 0 \xrightarrow{d_3} 1 \xrightarrow{\phi} 0 \xrightarrow{r_5} 0 \xrightarrow{d_3} 1$$

And these two strings are the only ones. They also only differ by swapping ϕd with $d\phi$, but their sequences of $D_S(-1)$ are identical, which is interesting.

Let's look at that example of 12367 that gave $D_S(-1) = 2$.



This is an L-semigroup, which makes me want to finish looking at them in general for $D_S(-1)$. But anyway, let's go through a few strings. One interesting thing to notice is that if we just build up the staircase $\{1,3,5\}$ with the strings we had before, we're stuck in having to then apply ϕ to try to get to $\{1,2,3,6,7\}$, which gives $\{1,2,4,6\}$, which isn't a numerical semigroup.

So we go a different route! This time, let's try to build up the entire first column first:

$$\{1\} \xrightarrow{\phi} \{1,2\} \xrightarrow{\phi} \{1,2,3\} \xrightarrow{\phi} \{1,2,3,4\} \xrightarrow{\phi} \{1,2,3,4,5\}$$

Then we go

$$\{1,2,3,4,5\} \xrightarrow{d_6} \{1,2,3,4,6\} \xrightarrow{\ell_5} \{1,2,3,5,6\} \xrightarrow{d_7} \{1,2,3,5,7\}$$

At this point, we place a single block at 6, but that makes me realize something goofy we forgot earlier, inner corners that aren't deep!

15.1 Corner Analysis Revamped

Ok, let's try to unify the notation a bit. I still like the d, ℓ, r because they are clearer than choosing ϕ_1, ϕ_2, ϕ_3 or something like that. So in this case, let's just define adding a regular inner corner by c. So our definitions are:

$$\ell_s = \text{adding a left-deep corner at } s$$

$$r_s = \text{adding a right-deep corner at } s$$

$$d_s = \text{adding a deep corner at } s$$

$$c_s = \text{adding a non-deep corner at } s$$

$$\phi = shift = d_0 \textit{ or } \ell_0$$

Then we have an association of numerical semigroups to the chains connecting them to $\{1\}$ via the above five operations, and let's call this set $str(S)$. Then our previously defined $\mathfrak{t}(S)$ is the size of $str(S)$.

I'll say it but won't actually do it at the moment: We do naturally have a graph describing this whose vertex set is $\mathcal{N}_\infty$ (the set of all numerical semigroups) by constructing a Cayley graph of sorts with these transition functions. I say of sorts because these operations don't form a group in any immediately obvious sense.

Well, I guess if we ignore the s, they do form a free group on 5 generators by default. And they do have interactions - but it's more like we'll have restricted substrings...and that means pattern avoidance! I've mentioned my work at an REU with Bruce Sagan, where he introduced this topic to us. He's absolutely a resource to look towards, as well as Richard Stanley, Lara Pudwell, Miklos Bona (who also wrote the book that really introduced me a unified look at combinatorics with his book **A Walk Through Combinatorics**), Toufik Mansour.

I worked with 4 others at the REU, with the help and guidance of Bruce Sagan and his graduate student Samantha Dahlberg, on Pattern avoidance in restricted growth functions. It was based on four statistics defined by Michelle

Wachs and Dennis White on *avoidance classes* for a fixed restricted growth function (which I'll define in a second). These avoidance classes and the resulting (sometimes multi-variate) generating functions have many wonderful connections to other areas of math or to each other through very interesting statistic-swapping bijections.

Let's take a quick dive into that. A *restricted growth function (RGF)* is a sequence of positive integers $w = w_1 w_2 \ldots w_n$ so that

1. $w_1 = 1$.
2. For $i \geq 2$, we have

$$w_i \leq 1 + max\{w_1, \ldots, w_{i-1}\}$$

And the name is clear from the second condition - the next element we add to a sequence cannot increase from the previous maximum value by more than 1. For example, $w = 11213$ is an RGF, while $w = 123325$ is not because $5 > 3 + 1$.

If we have a restricted growth sequence that doesn't start at 1, we can standardize it by replacing the smallest element by 1, the second smallest with 2, and so on. Here are a couple of examples:

$$st(25332) = 13221$$

$$st(1678) = 1234$$

We also have a bijection to set partitions (which I won't go into here), so we find we can study set partitions via RGFs!

Moving on, if we fix an RGF v, which we'll call the pattern, then we say that w avoids v if there is no subsequence w' of w so that $st(w') = v$. We call the set of all RGFs of length n R_n, and the big object of study is the avoidance class $R_n(v)$ consisting of RGFs of length n that avoid v. If desired, we can look at avoidance classes of multiple patterns.

A nice first example is $v = 12$. This means we can't have some $w' = w_1 w_2$ so that $st(w_1 w_2) = 12$. All this means is that we can't have $w_1 < w_2$, so we can't have any increases, so

$$R_n(12) = \{1111 \ldots 1\}$$

consists of a single RGF. Sagan classified $R_n(v)$ for all patterns of length 3, which I'll list here because it's a very nice result:

$$R_n(111) = \{w \in R_n : \textit{ every element of } w \textit{ appears at most twice}\}$$

$$R_n(112) = \{w \in R_n : w \textit{ has initial run } 12 \ldots m \textit{ and } m \geq w_{m+1} \geq \cdots \geq w_n\}$$

$$R_n(121) = \{w \in R_n : w \textit{ is weakly increasing}\}$$

$$R_n(122) = \{w \in R_n : \textit{every element } j \geq 2 \textit{ of } w \textit{ appears at most once}\}$$

$$R_n(123) = \{w \in R_n : w \textit{ contains only } 1s \textit{ and } 2s\}$$

Many patterns of longer length have been studied, but are much harder to characterize. But instead of that, we looked at four statistics by Wachs and White for RGFs, lb, ls, rb, rs. We'll define $lb(w)$ and the others will be defined the exact same way. For a word $w = w_1 w_2 \dots w_n$, we define

$$lb(w_j) = \#\{w_i \ : \ i < j \textit{ and } w_i > w_j\}$$

This means $lb(w_j)$ is the number of elements to the left of w_j that are bigger than w_j. Then

$$lb(w) = lb(w_1) + lb(w_2) + \cdots + lb(w_n)$$

And similarly

$$lb = left\ bigger$$

$$ls = left - smaller$$

$$rb = right - bigger$$

$$rs = right - smaller$$

All four just mean we choose a point, look in a direction, and check if they're smaller or bigger. Then we have the individual statistic generating function

$$LB_n(v) = \sum_{w \in R_n(v)} q^{st(w)}$$

and similarly defined for others.

There are connections to all different areas of combinatorics, but I'll just show the most basic (and the first!) complete characterization:

Theorem 11 *We have*

$$RS_n(122) = LB_n(123) = RS_n(123) = 1 + \sum_{k=0}^{n-2} \binom{n-1}{k+1} q^k$$

Another theorem I'll include is the characterization of an avoidance class of two RGFs:

Theorem 12 *We have*

$$R_n(112, 1221) = \{12 \dots m k^{n-m} \ : \ 1 \leq k \leq m\}$$

Fun refresher! It makes me think that the gaps of a numerical semigroup S of genus g look kind of like a RGF of length g, in the sense that their growth is certainly restricted. If we have a set $\{a_1, \dots, a_t\}$ of gaps in increasing order, then we have a weak bound that $a_i < 2(a_{i-1} + 1)$.

Obviously not everything has to be connected but I have a general idea I want to look into:

1. Construct a map from $\mathcal{N}_g$ to RGFs.

2. Find an "approximation" of $\mathcal{N}_g$ by pattern avoidance classes.

The gaps themselves are clearly not RGFs as defined in the previous part. And the difference set won't work, because of - for example - 1236, which would map to 113, not an RGF. As we've learned from past examples, we should probably take into account the additive structure of S. And obviously, two sets being the same size doesn't mean there has to be some meaningful bijection between the two. But it never hurts to look!

Let's try one based off the fragile gap stuff. We'll define a word $w = w_1 w_2 \dots w_n$ from a numerical semigroup S by

$$w_i = \#\{0 \leq j < i \ : \ j + a_i \in S^c\}$$

Since $a_1 = 1$, we have $w_1 = \#\{0 \leq j < 1 \ : \ j + 1 \in S^c\} = \#\{0\} = 1$. On the flip side, $w_g = \#\{0 \leq j < g \ : \ j + F(S) \in S^c\} = \#\{0\} = 1$.

Let's do some examples:

$$w(\{1, 2\}) = 11$$

$$w(\{1, 3\}) = 11$$

Ok, overlap already...let's do some general examples. Take $S = \mathcal{O}_g$, the ordinary semigroup of genus g. Then

$$w_i = \#\{0 \leq j < i \ : \ j{+}i \in S^c\} = \#\{0 \leq j < i \ : \ j{+}i \leq g\} = \#\{0 \leq j < i \ : \ j \leq g{-}i\} =$$

$$= min(i, g - i + 1)$$

For example, taking $g = 5$, we have $\mathcal{O}_5 = \{1, 2, 3, 4, 5\}$, and

$$w_1 = \#\{0\} = 1$$

$$w_2 = \#\{0, 1\} = 2$$

$$w_3 = \#\{0, 1, 2\} = 3$$

$$w_4 = \#\{0, 1\} = 2$$

$$w_5 = \{0\} = 1$$

And an even example of $g = 6$ gives

$$w_1 = \#\{0\} = 1$$

$$w_2 = \#\{0, 1\} = 2$$

$$w_3 = \#\{0, 1, 2\} = 3$$

$$w_4 = \#\{0, 1, 2\} = 3$$

$$w_5 = \{0, 1\} = 2$$

$$w_6 = \{0\} = 1$$

That's really cool! Under this map, we'd get

$$w(\mathcal{O}_g) = \begin{cases} 12\ldots(\frac{g+1}{2}-1)\left(\frac{g+1}{2}\right)(\frac{g+1}{2}-1)\ldots 21 & g \text{ odd} \\ 12\ldots\left(\frac{g}{2}-1\right)\left(\frac{g}{2}\right)\left(\frac{g}{2}\right)\left(\frac{g}{2}-1\right)\ldots 21 & g \text{ even} \end{cases}$$

So we get a different symmetry than the one seen with the partition. We get a unimodal RGF. How about the staircase partition $\{1, 3, \ldots, 2g-1\}$? Then

$$w_i = \#\{0 \le j < i \ : \ j + 2i - 1 \textit{ is odd and } \le \ 2g - 1\}$$

$$\#\{0 \le j < i \ : \ j \textit{ is even and } \le \ 2(g-i)\}$$

Let's do some examples again. For $S = \{1, 3, 5\}$, we have

$$w_1 = \#\{0\} = 1$$

$$w_2 = \#\{0\} = 1$$

$$w_3 = \#\{0\} = 1$$

Ok, maybe a bigger example, $S = \{1, 3, 5, 7, 9, 11, 13\}$,

$$w_1 = \#\{0\} = 1$$

$$w_2 = \#\{0\} = 1$$

$$w_3 = \#\{0, 2\} = 2$$

$$w_4 = \#\{0, 2\} = 2$$

$$w_5 = \#\{0, 2, 4\} = 3$$

$$w_6 = \#\{0, 2\} = 2$$

$$w_7 = \#\{0\} = 1$$

For an even example, let's do $S = \{1, 3, 5, 7, 9, 11\}$,

$$w_1 = \#\{0\} = 1$$

$$w_2 = \#\{0\} = 1$$

$$w_3 = \#\{0, 2\} = 2$$

$$w_4 = \#\{0, 2\} = 2$$

$$w_5 = \#\{0, 2\} = 2$$

$$w_6 = \#\{0\} = 1$$

So in general, we will have

$$w(\{1, 3, 5, \ldots, 2g{-}1\} = \begin{cases} 1^2 2^2 \ldots \left(\frac{g+1}{2}-1\right)^2\left(\frac{g+1}{2}\right)\left(\frac{g+1}{2}-1\right)\ldots 21 & g \text{ odd} \\ 1^2 2^2 \ldots \left(\frac{g}{2}-2\right)^2\left(\frac{g}{2}-1\right)^3\left(\frac{g}{2}-2\right)\left(\frac{g}{2}-3\right)\ldots 21 & g \text{ odd} \end{cases}$$

Will it always be unimodal? I wonder if the symmetry of the word is related to whether the numerical semigroup is symmetric? I guess not, since $w(\{1,2,3,7\}) = 1211$.

Let's choose a random one: $S = \{1,2,3,4,7,8,13\}$

$$w_1 = \#\{0\} = 1$$

$$w_2 = \#\{0,1\} = 2$$

$$w_3 = \#\{0,1\} = 2$$

$$w_4 = \#\{0,3\} = 2$$

$$w_5 = \#\{0\} = 1$$

$$w_6 = \#\{0\} = 1$$

Let's do another, $S = \{1,2,3,6,7\}$:

$$w_1 = \#\{0\} = 1$$

$$w_2 = \#\{0,1\} = 2$$

$$w_3 = \#\{0\} = 1$$

$$w_4 = \#\{0,1\} = 2$$

$$w_5 = \#\{0\} = 1$$

SO $w(\{1,2,3,6,7\}) = 12121$, awesome! Breaking unimodality and being the first numerical semigroup to have $D_S(-1) \neq 0, 1$, it seems to be a good example to keep in mind!

Ok, I'm interested enough to go ahead and enumerate these by genus:

S^c	$w(S)$	$\#distinct$	$N(g)$
$\{1\}$	1	1	1
$\{1,2\}$	11	1	2
$\{1,3\}$	11		
$\{1,2,3\}$	121	2	4
$\{1,2,4\}$	111		
$\{1,2,5\}$	111		
$\{1,3,5\}$	111		
$\{1,2,3,4\}$	1221	4	7
$\{1,2,3,5\}$	1221		
$\{1,2,3,6\}$	1211		
$\{1,2,3,7\}$	1211		
$\{1,2,4,5\}$	1121		
$\{1,2,4,7\}$	1111		
$\{1,3,5,7\}$	1121		

So a fair bit of overlap but it might just be because we haven't given them enough space (i.e. high enough genus).

What I want to try to do now is show this is truly an RGF. We know $w_1 = 1$, so we just need to show that for all $i \geq 2$,

$$w_i \leq 1 + max\{w_1, \dots, w_{i-1}\}$$

And w_i is the number of j less than i for which $j + a_i \in S^c$. For a basic failure, let's look for a $1 \dots 13$. For example, 113 would mean $1 \in S^c$ and $2 \notin S^c$. Which means we must have a staircase semigroup, which starts 112.

What if it occurs further down the word?

$$S = \{1, 2, 3, 4, 5, 6, 7, 9, 11, 12, 14, 15, 17\}$$

$$w(S) = 1234444554321$$

I see another pattern that helps see what this really is - the final string is always a decreasing sequence after we pass a certain point. And that's because we can think of w_i as being the number of gaps in the interval $[a_i, a_i + i)$.

So how does the interval change from one gap to the next? Its length is i, so the length increases by 1. Any gaps in the interval $[a_{i+1}, a_i + i)$ will also be counted by w_i, with one missing because we no longer count a_i. This means that given enough space,

$$w_{i+1} = w_i - 1 + \#\{a \in S^c \ : \ a_i + i \leq a < a_{i+1} + i + 1\}$$

And by "given enough space", I mean $a_{i+1} + i + 1 \leq F(S)$. Let's go ahead and define the *excess gap index* of S to be the largest i so that $a_i + i \leq F(S)$. We'll non-creatively call this $egi(S)$. Then $w(S)$ will end in a decreasing chain after $egi(S)$, since the set $\{a \in S^c \ : \ a_i + i \leq a < a_{i+1} + i + 1\}$ will be empty. Here's a few examples, with excess gap colored in blue.

$$\{1\}$$

$$\{1, 2\}$$

$$\{1, 3\}$$

$$\{1, 2, 3\}$$

$$\{1, 2, 4\}$$

$$\{1, 2, 5\}$$

$$\{1, 3, 5\}$$

Let's jump to some larger examples.

$$\{1, 2, 3, 4, 5, 6, 7, 8\}$$

$$\{1, 2, 3, 4, 5, 7, 10\}$$

$$\{1,2,3,4,5,7,10,13\}$$

Since we know that $w_g = 1$, this puts a bound on how large $egi(S)$ can get.

For a numerical semigroup S to map to a non-RGF, we'd need

$$\#\{a \in S^c \ : \ a_i + i \leq a < a_{i+1} + i + 1\} \geq 3$$

This doesn't seem impossible. On the other hand, it seems like a large L-semigroup might achieve this. Let's try...

$$S = \{1,2,3,4,5,6,7,8,15,16,17\}$$

$$w_i = i \qquad 1 \leq i \leq 4$$

$$w_5 = \#\{0,1,2,3\} = 4$$

$$w_6 = \#\{0,1,2\} = 3$$

$$w_7 = \#\{0,1\} = 2$$

$$w_8 = \#\{0,7\} = 2$$

$$w_9 = \#\{0,1,2\} = 3$$

$$w_{10} = \#\{0,1\} = 2$$

$$w_{11} = \#\{0\} = 1$$

So that didn't work out. Let's try a different way:

$$S = \{1,2,3,4,5,10,11\}$$

$$w_1 = 1$$

$$w_2 = 2$$

$$w_3 = 3$$

$$w_4 = 2$$

$$w_5 = 1$$

$$w_6 = 2$$

$$w_7 = 1$$

Still nothing, so maybe that's a good sign! Let's take a moment to find $egi(S)$ for a range of numerical semigroups. It feels related to the depth/alpha-dimension/Frobenius number. For a general example, if we have $\mathcal{O}_g$, then $a_i + i \leq F(S)$ means $i + i \leq g$, so

$$egi(\mathcal{O}_g) = \left\lfloor \frac{g}{2} \right\rfloor$$

This explains the strict unimodal form of $w(S)$: we get a forced increasing part from the initial sequence $1, 2, \ldots, m(S) - 1$ of gaps, and then a forced decreasing part after $egi(S)$. For $\mathcal{O}_g$, these overlap, so we get the perfect rise and fall, with the end of the rise and beginning of the fall being the same when g is even.

I'm also going to go ahead and conjecture that $\mathcal{O}_g$ minimizes $egi(S)$ for a fixed genus, probably similar to our argument for depth where we show that moving elements further apart weakly increases $egi(S)$. But I'll try that in a minute.

And on the other end, the staircase semigroup would have $2i-1+i \leq 2g-1$, so $3i \leq 2g$, which finally gives

$$egi(\{1,3,\ldots,2g-1\}) = \left\lfloor \frac{2g}{3} \right\rfloor$$

Coding this up is very easy, you just have to account for the indexing starting at 0 instead of 1.

```
def egi(S):
    for i in range(len(S)):
        if S[i] + i+1 > max(S):
            return(i)
            break
```

We'll include $F(S)$ as well.

S	$egi(S)$	$F(S)$
$\{1\}$	1	1
$\{1,2\}$	1	2
$\{1,3\}$	1	3
$\{1,2,3\}$	1	3
$\{1,2,4\}$	2	4
$\{1,2,5\}$	2	5
$\{1,3,5\}$	2	5
$\{1,2,3,4\}$	2	4
$\{1,2,3,5\}$	2	5
$\{1,2,3,6\}$	3	6
$\{1,2,3,7\}$	3	7
$\{1,2,4,5\}$	2	5
$\{1,2,4,7\}$	3	7
$\{1,3,5,7\}$	2	7
$\{1,2,3,4,5\}$	2	5
$\{1,2,3,4,6\}$	3	6
$\{1,2,3,4,7\}$	3	7
$\{1,2,3,4,8\}$	4	8
$\{1,2,3,4,9\}$	4	9
$\{1,2,3,5,6\}$	3	6
$\{1,2,3,5,7\}$	3	7
$\{1,2,3,5,9\}$	4	9
$\{1,2,3,6,7\}$	3	7
$\{1,2,4,5,7\}$	3	7
$\{1,2,4,5,8\}$	3	8
$\{1,3,5,7,9\}$	3	9

And let's go ahead and write out the stat generating function, because I always find those visually helpful. I'll include $g = 6, 7, 8$ too.

$$EGI(1) = x$$

$$EGI(2) = 2x$$

$$EGI(3) = x + 3x^2 = x(1 + 3x)$$

$$EGI(4) = 4x^2 + 3x^3 = x^2(4 + 3x)$$

$$EGI(5) = x^2 + 8x^3 + 3x^4 = x^2(1 + 8x + 3x^2)$$

$$EGI(6) = 5x^3 + 15x^4 + 3x^5 = x^3(5 + 15x + 3x^2)$$

$$EGI(7) = x^3 + 16x^4 + 19x^5 + 3x^6 = x^3(1 + 16x + 19x^2 + 3x^3)$$

$$EGI(8) = 7x^4 + 31x^5 + 26x^6 + 3x^7 = x^4(7 + 31x + 26x^2 + 3x^3)$$

There are a few patterns that seem to pop out already. I'll make a list of (admittedly weak with little data) conjectures, and then see if we can prove them:

Conjecture 12 *For $g \geq 3$, we have*

1. $x^{\lfloor \frac{g}{2} \rfloor} || EGI(g)$
2. *The degree of $EGI(g)$ is $g-1$ and the leading coefficient is always* 3.
3. *The number of terms in $EGI(g)$ is $\lceil \frac{g}{2} \rceil$*
4. *The coefficients of $EGI(g)$ are unimodal.*
5. *For odd g, there is a unique numerical semigroup minimizing $egi(S)$. For even g, there is an increasing number of semigroups minimizing $egi(S)$.*

For Conj Part 1 and Conj Part 2, we need to prove what we discussed before: Show that $\mathcal{O}_g$ minimizes $egi(S)$ for a fixed genus. And hopefully in doing so, we get the more detailed parts too.

So as a next step to $\mathcal{O}_g$, let's see what the $L_{f,g}$ semigroups give us. They are of the form

$$L_{f,g}^c = \{1, 2, \ldots, g-1, f\}$$

so we want the largest i so that $i + i \leq f$, which means

$$egi(L_{f,g}) = \left\lfloor \frac{f}{2} \right\rfloor$$

Awesome, I think this gives our range as well (Conj Part 3). For a fixed g, we can have $g+1 \leq f \leq 2g-1$, so our egi ranges from

$$\left\lfloor \frac{g+1}{2} \right\rfloor \leq egi(L_{f,g}) \leq \left\lfloor \frac{2g-1}{2} \right\rfloor$$

The upper bound is $\lfloor g - \frac{1}{2} \rfloor = g-1$. And that's Conj Part 2 and Conj Part 3. Why should there only be three of them? For $g = 5$, we have

$$\{1, 2, 3, 4, 8\}, \{1, 2, 3, 4, 9\}, \{1, 2, 3, 5, 9\}$$

For $g = 6$, we have

$$\{1, 2, 3, 4, 5, 10\}, \{1, 2, 3, 4, 5, 11\}, \{1, 2, 3, 4, 6, 11\}$$

For $g = 7$, we have

$$\{1, 2, 3, 4, 5, 6, 12\}, \{1, 2, 3, 4, 5, 6, 13\}, \{1, 2, 3, 4, 5, 7, 13\}$$

So we definitely have a pattern:

$$L_{2g-1,g} \quad L_{2g-2,g} \quad \{1, 2, \ldots, g-2, g, 2g-1\}$$

The last one is a numerical semigroup, since $2(g-1)=2g-2$ and $g-1+g+1=2g$, and all other sums are larger than $2g$. So proving these are the only three answers Conj Part 2.

So what about Conj Part 4 and Conj Part 5? Well I want to say that

$$egi(S) \leq \left\lfloor \frac{F(S)}{2} \right\rfloor$$

so let's go ahead and update that table from before with $F(S)$. And it's surprising just how often the bound actually equals $egi(S)$! But it is sometimes off by 1 and probably more as the genus grows.

But anyway, let's save this for later and go back to working with $w(S)$. We have the recurrence

$$w_{i+1} = w_i - 1 + \#\{a \in S^c \ : \ a_i + i \leq a < a_{i+1} + i + 1\}$$

which is what tells us that we get a decreasing chain after $egi(S)$. But before that point, it tells us about the change in w_i. Let's do an example with some numerical semigroups we previously worked with.

$$S = \{1, 2, 3, 4, 5, 10, 11\}$$
$$w_1 = 1$$
$$w_2 = 2 \qquad [2, 4) = \{2, 3\}$$
$$w_3 = 3 \qquad [4, 6) = \{4, 5\}$$
$$w_4 = 2 \qquad [6, 8) = \{\}$$
$$w_5 = 1 \qquad [8, 10) = \{\}$$
$$w_6 = 2$$
$$w_7 = 1$$

Let's try one with a bit more spread (higher α-dimension):

$$S = \{1, 2, 4, 5, 7, 8, 10\}$$
$$w_1 = 1$$
$$w_2 = 1 \qquad [2, 4)$$
$$w_3 = 2 \qquad [4, 7)$$
$$w_4 = 3 \qquad [7, 9)$$
$$w_5 = 3 \qquad [9, 12)$$
$$w_6 = 2 \qquad [12, 14)$$
$$w_7 = 1 \qquad [14, 17)$$

Maybe even larger?

$$S = \{1, 2, \ldots, 30, 45, 46, 47, 48, 49, 50, 59, 60\}$$
$$w_1 = 1$$
$$w_2 = 2$$
$$\vdots$$
$$w_{15} = 15$$
$$w_{16} = 15 \qquad [30, 32)$$
$$w_{17} = 14 \qquad [32, 34)$$
$$\vdots$$
$$w_{22} = 8 \qquad [42, 44)$$
$$w_{23} = 8 \qquad [44, 46)$$
$$w_{24} = 9 \qquad [46, 48)$$
$$w_{25} = 10 \qquad [48, 50)$$
$$w_{26} = 10 \qquad [50, 52)$$
$$w_{27} = 9 \qquad [52, 54)$$
$$w_{28} = 8 \qquad [54, 56)$$
$$w_{29} = 7 \qquad [56, 58)$$
$$w_{30} = 7 \qquad [58, 60)$$
$$w_{31} = 7 \qquad [60, 76)$$
$$w_{32} = 6$$
$$\vdots$$

I feel I must be missing something obvious. To get a large interval, we must have a large separation of gaps. But then we won't pick up many gaps in the interval, so w_i is changed by at most one? As previously stated, the length of the interval is
$$a_{i+1} + i + 1 - (a_i + i) = a_{i+1} - a_i + 1$$
but that's exactly the distance between the two gaps a_i and a_{i+1}, so we're just looking at an interval $[a_i, a_{i+1}]$ shifted up by i.

$$S = \{1, 2, 3, 4, 7, 8, 9\}$$
$$w_1 = 1$$
$$w_2 = 2$$
$$w_3 = 2$$

$$w_4 = 2$$
$$w_5 = 3$$
$$w_6 = 2$$
$$w_7 = 1$$

So if we can get a large interval $[a_i, a_{i+1}]$ which contains a lot of S^c when shifted by i, then we can do this. But shifting by i is a pretty big shift, so a large interval would prevent, for example,

$$2(a_i + 1), 2a_i + 3, 2a_i + 4, \ldots, a_i + a_{i+1}, \ldots, 2(a_{i+1} - 1)$$

from being gaps in S further on.

As $a_i \geq i$, these are at least

$$a_i + i + 2, a_i + i + 3, a_i + i + 4, \ldots, a_i + a_{i+1}, \ldots, a_{i+1} + i - 1, a_{i+1} + i$$

But the interval we're intersecting with is $[a_i + i, a_{i+1} + i + 1)$, which highly intersects the forbidden interval above! The only extra spaces we get are

$$a_i + i \quad \textit{and} \quad a_i + i + 1$$

Ok, let's see if we can test this.

$$S = \{1, 2, 4, 5, 7, 8, 10\}$$

$$w_1 = 1$$
$$w_2 = 1 \qquad [2, 4) = \{2\}$$
$$w_3 = 2 \qquad [4, 7) = \{4, 5\}$$
$$w_4 = 3 \qquad [7, 9) = \{7, 8\}$$
$$w_5 = 3 \qquad [9, 12) = \{10\}$$
$$w_6 = 2 \qquad [12, 14)$$
$$w_7 = 1 \qquad [14, 17)$$

To summarize this, I think we can show that w_i changes by $-1, 0, 1$ depending on how many of $a_i + i,\ a_i + i + 1$ are gaps of S. Which actually seems kind of crazy, because that means it's telling us about when consecutive pairs are GAP-GAP, GAP-ELEMENT, or ELEMENT-GAP, which is what we were looking at with the symmetric difference $S \triangle (1 + S)$ before.

I'll call a sequence of integers an *integer-continuous function (ICF)* if it is as continuous as a discrete function can be: For all k,

$$|f(k + 1) - f(k)| \leq 1$$

I did some statistic-type stuff with ICF's in the past, so maybe I'll go into that later. I have found this referenced in a blog post and a very interesting MAA note. Also, it's not too hard a counting problem to see that if we fix $f(1) = 0$, then we have 3^{n-1} ICFs of length n.

So now we've arrived upon the following theorem:

Conjecture 13 *The map sending a numerical semigroup S to its word $w(S) = w_1 \ldots w_g$ given by*

$$w_i = \#\{0 \leq j < i \ : \ j + a_i \in S^c\}$$

is a map into the set of ICFs of length g.

I don't want to try to write out a formal proof right now, so I'll label it as a conjecture. But we really lost the thread here! You'll recall on page 123, we began to try to analyze strings of block-building (which makes it sound like we're kindergarteners) to get to a numerical semigroup.

16 Corner Analysis Take 2

I'll go ahead and directly copy and paste from the last one:

$$\ell_s = \text{adding a left-deep corner at } s$$

$$r_s = \text{adding a right-deep corner at } s$$

$$d_s = \text{adding a deep corner at } s$$

$$c_s = \text{adding a non-deep corner at } s$$

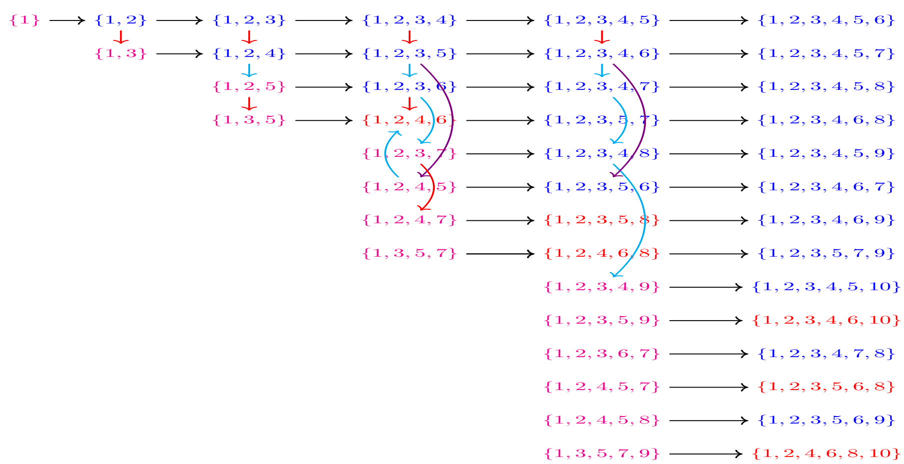
$$\phi = shift = d_0 \text{ or } \ell_0$$

Then we have an association of numerical semigroups to the chains connecting them to $\{1\}$ via the above five operations, and let's call this set $str(S)$. And our previously defined $\mathfrak{t}(S)$ is the size of $str(S)$. We'll write them without the subscript for now. And unlike last time, I want to actually look at that (directed,semi-) Cayley graph with vertex set $\mathcal{N}_\infty$ and generators $\{\ell, r, d, c, \phi\}$.

This is what we previously called $\mathcal{B}$, but with edge labels specifying what type of corner block we add.

$$\begin{tikzcd}
 & & & \{1\} \arrow[dl, "\phi"'] \\
 & & \{1,2\} \arrow[dl, "\phi"'] \arrow[dr, "d"] & \\
 & \{1,2,3\} \arrow[dl, "\phi"'] \arrow[dr, "d"] & & \{1,3\} \arrow[dl, "\phi"] \\
\{1,2,3,4\} & & \{1,2,4\} &
\end{tikzcd}$$

And this makes me realize, with the goal of understanding shifted/non-shifted sets, let's copy the forest of chains from before (around page 41) and connect sets based on ℓ, r, d, c, color-coded. And for ease of drawing the partitions, I use this pattern-designing website for crafts.



So one thing we see is that $\{1,3,5,7\}$ can't be reached even with these operations, and is "primitive" in this graph. And the reason is that while $\{1,2,4,7\}$ is closest, we must add two blocks to get to $\{1,3,5,7\}$. Which leads to the following question:

Question 7 *Suppose we allow adding up to k blocks and look at the resulting digraph $\mathcal{B}^k$. Distances between numerical semigroups will of course get smaller the larger k goes. But does there exist some absolute k for which $deg_{in}(S) > 0$ for all $S \in \mathcal{N}_\infty$? Or do the staircase semigroups get arbitrarily far away from any other numerical semigroup?*

Really, I guess this almost doesn't really take into account being numerical semigroups. Just n_S or $\pi(S)$, which we can compute with

```
for S in V5:
    print(S)
    A = 0
    for i in range(max(S)):
        if i not in S:
            pg = [x for x in S if x > i]
            A += len(pg)
    print(A)
    print('-------')
```

And for $g = 2, 3, \ldots$, we get

$$3 - 2 = 1$$

$$6 - 5 = 1$$

$$10 - 8 = 2$$

$$15 - 10 = 5$$

$$21 - 16 = 5$$

$$28 - 21 = 7$$

$$36 - 25 = 11$$

It seems this doesn't appear in the OEIS, but there are some close sequences related to partitions. The first one, for the staircase partition, is clearly $\binom{g+1}{2}$. I wonder what the second highest numerical semigroups tend to look like? Here are the examples for $g \geq 2$:

$$\{1, 2\}$$

$$\{1, 2, 5\}$$

$$\{1, 2, 4, 7\}$$

$$\{1, 2, 3, 5, 9\} \quad or \quad \{1, 2, 4, 5, 8\}$$

$$\{1, 2, 4, 5, 8, 11\}$$

$$\{1, 2, 4, 5, 7, 10, 13\}$$

$$\{1, 2, 3, 5, 7, 9, 11, 15\}$$

On the other hand, if we remove the staircase partitions, then we seem to have very close semigroups, like differing by 1 block. So perhaps $\mathcal{B}^k$ does have the desired property if we remove $\{1, 3, \ldots, 2g - 1\}$. Let's look at some intervals contained in

$$\{n_S \ : \ S \in \mathcal{N}_g\}$$

For $g = 1$, we have $[1]$ and for $g = 2$, we have $[2, 3]$.
For $g = 3$, we have

$$[3, 4, 5, 6]$$

For $g = 4$, we have

$$[4, 5, 6, 7, 8]$$

For $g = 5$, we have

$$[5, 6, 7, 8, 9, 10]$$

For $g = 6$, we have

$$[6, 7, 8, 9, 10, 11, 12] \quad and \quad [14, 15, 16]$$

For $g = 7$, we have

$$[7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]$$

For $g = 8$, we have

$$[8, 9, 10, 11, 12, 13, \ldots, 20, 21, 22, 23, 24, 25]$$

And in a lot of these cases, the sole exclusion is the staircase semigroup. So perhaps if we ignore this outlier, we can still achieve our goals! And I guess the fact that it's the only semigroup starting with $\{1, 3\}$ makes it an outlier anyway.

Looking at these intervals shows us that we at least need to be able to add 2 blocks at a time, since $g = 6$ is missing 13.