

GBP—HAMILTONICITY—SUMMER 2018

GBP

A note to the reader: all graphs here are simple, connected, and of order at least three.

SOME TRUE CONJECTURES

Observation 1. *If G is a complete graph, then G is Hamiltonian.*

Lemma 1. *If G is a bipartite k -regular graph, with $k > 0$ and bipartition (X, Y) , then $|X| = |Y|$.*

This is immediate by double counting; $k|X| = \sum_{x \in X} d(x) = e(G) = \sum_{y \in Y} d(y) = k|Y|$.

Lemma 2. *If there is a graph which is (n, k, λ, μ) -strongly-regular, then $(n - k - 1)\mu = k(k - \lambda - 1)$.*

Proof. We partition the graph into three sets. Let x be an arbitrary vertex; we think of this as the root of our partition; say $P_0 = \{x\}$. Since G is k -regular, we think of its k neighbors as being in the second set; formally, take $P_1 = \{y : y \in N(x)\}$. All other nodes form the third set; take $P_2 = \{y : y \notin N(x) \wedge y \neq x\}$.

Since the vertices of P_1 are adjacent to x , they have λ neighbors in common with the x ; these vertices must also lie in P_1 . Since our graph is k -regular, there are $(k - \lambda - 1)$ edges remaining from each P_1 vertex to P_2 . Thus we have $e(P_1, P_2) = k(k - \lambda - 1)$.

Since the vertices in P_2 are not adjacent to x , they must have μ common neighbors with x ; these vertices are in P_1 . Since $|P_2| = (n - k - 1)$, and each vertex of P_2 is connected to μ nodes of P_1 , we have $e(P_1, P_2) = (n - k - 1)\mu$. Comparing with our earlier count of $e(P_1, P_2)$, we see that $(n - k - 1)\mu = k(k - \lambda - 1)$. \square

Theorem 3. *If G is a bipartite, connected, (n, k, λ, μ) -strongly-regular graph, then G is a complete balanced bipartite graph.*

Proof. Assume that G is as stated, with bipartition (X, Y) .

First, notice that if $\lambda > 0$, then G contains a triangle; this is impossible, if G is bipartite, and so we may assume that $\lambda = 0$.

Now, assume that $\mu = 1$; this implies that G is a Moore graphs with girth five; this is impossible as G is purported to be bipartite.

Next, suppose that $\mu = 0$. A quick counting argument shows that always $(n - k - 1)\mu = k(k - \lambda - 1)$. Simplifying in the case that $\mu = 0$ and $\nu = 0$, we see that $k - 1 = 0$, and so $k = 2$. Thus G is an even cycle, which is never

strongly regular with the given parameters.

Finally, suppose that $\mu \geq 2$, and consider $x \in X$ and $y \in Y$ such that $xy \notin E(G)$. Then by definition these vertices have $\mu \geq 2$ neighbors in common; this is impossible as G is bipartite and x, y lie in different parts. G can have no such non-edge, and so it is complete. By our lemma, it is also balanced since it is regular. \square

Observation 2. *If G is a line graph whose complement is chordal and which has radius equal to diameter, then G is Hamiltonian.*

“is-line-graph implies claw-free; is-complement-of-chordal implies P6-free; has-radius-equal-diameter implies 2-connected; And every 2-connected (P6 ,claw)-free graph is hamiltonian”

Observation 3. *If G is a two connected outerplanar graph, then it is Hamiltonian.*

This is trivial; the boundary of the infinite face must thus be a cycle, and as every vertex is on this face it forms a spanning cycle.

Observation 4. *If G is a Gallai tree and is two connected, then G is Hamiltonian.*

Again, this is immediate – any two bricks in a Gallai tree are separated by a cut vertex, so a two connected Gallai tree must be just a single brick. This is either an odd cycle or a complete graph, so it is Hamiltonian.

Lemma 4. *If G is a nontrivial cartesian product and outerplanar, then it is Hamiltonian.*

This relies on the fact that a graph is outerplanar if and only if it contains no $\{K_{2,3}, K_4\}$ -minor.

Observation 5. *If G is a graph with its diameter equal to its radius, and G is circular planar, then it is Hamiltonian.*

This is again trivial; if G contains a cut vertex or cut edge, and is outerplanar, then it has its diameter larger than its radius.

Observation 6. *If G is a bipartite line graph, and it has diameter equal to radius, then it is Hamiltonian.*

Recall that a graph is a line graph if and only if it can be partitioned into edge disjoint cliques such that every vertex lies in exactly two cliques. Since G is bipartite, it is triangle free, so such cliques are singletons or edges. Every vertex is in exactly two such cliques, and so every vertex has degree at most two. Since G is connected, our graph is either a path or a cycle. Paths consisting of more than a single edge do not have radius equal to diameter, so our graph is a spanning cycle.

Lemma 5. *A graph is outerplanar if and only if it is $K_{2,3}$ -minor free and K_4 -minor-free.*

Lemma 6. $P_2 \square P_k$ is Hamiltonian for $k \geq 2$.

Lemma 7. *Suppose a graph H is connected, outerplanar, and Hamiltonian. Further suppose $H = H_1 \square H_2$ where H_1 and H_2 are graphs on at least two vertices.*

Proof. Since H is connected, so are H_1 and H_2 .

Claim 1. H is $P_2 \square P_m$

Suppose that H_1 has a cycle. Since H_2 has an edge, H contains $K_3 \square P_2$ as a minor and thus it contains $K_{2,3}$ as a minor (take the open neighborhood of any vertex to be the class of size 3), a contradiction.

Thus, H_1 and H_2 are trees. If H_1 has a vertex of degree 3 or more, then H contains $K_{1,3} \square P_2$ as a minor. Taking the neighborhood of one of the vertices of degree 3 in $K_{1,3}$ and the other copy of $K_{1,3}$ as the class of size 2, we see that H has $K_{2,3}$ as a minor, a contradiction.

Thus, H_1 and H_2 are paths. Suppose that both have three or more vertices. Thus H contains $H_1 \square H_2$ as a minor. Taking the neighbors of a vertex of degree 3 as the class of size 3 shows that H contains a $K_{2,3}$ -minor, a contradiction.

Thus at least one of H_1 and H_2 has exactly two vertices and the proof is complete. \square

PLANAR TRANSITIVE GRAPHS ARE HAMILTONIAN

Definition 8. *A graph automorphism is an isomorphism between the graph and itself. That is, it is a map $\phi : V(G) \rightarrow V(G)$ such that $xy \in E(G)$ iff $\phi(x)\phi(y) \in E(G)$. A graph G is vertex transitive if for every pair of vertices $u, v \in V(G)$, there is a graph automorphism ϕ with $\phi(u) = \phi(v)$.*

Observation 7. *Every vertex transitive graph is d -regular for some $k \in \mathbb{N}$.*

Theorem 9. [Mad70] *If G is a d -regular vertex transitive graph with connectivity k , then $\frac{2(d+1)}{3} \leq k$.*

Theorem 10. [Tut56] *Every 4-connected planar graph is Hamiltonian.*

Theorem 11. [Zel77] *The only 3-regular vertex transitive simple planar graphs are the tetrahedron, the dodecahedron, the n -sided prisms (for $n \geq 3$), the tricone graph, the truncated cube, truncated octahedron, truncated dodecahedron, truncated icosahedron (bucky ball), and the Great Rhombicosidodecahedral Graph.*

We prove the following theorem.

Theorem 12. *Every vertex transitive planar graph is Hamiltonian.*

By Theorem 10, it is enough for us to consider graphs with connectivity at most three. By Theorem 9, such graphs are regular of degree at most $7/2$; that is, they either have all vertices of degree two or of degree three. Since a two-regular connected graph is a cycle (and thus Hamiltonian), we focus only on the three regular case. But, these graphs are characterized by Theorem 11. The only infinite family of graphs here are the prisms (which are trivially Hamiltonian – just walk around all but a single edge on one of the end polygons, across the adjacent face, around the opposing end, and back along the same face). The other graphs are easily tested to be Hamiltonian in Sage.

REFERENCES

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