Introduction to Ordinary Differential Equations

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Outline

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2. Initial Value Problems
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   - Second order differential equations
   - Recasting high order differential equations as a system of first order differential equations
3. Boundary Value Problems
4. Solution techniques for nonlinear differential equations
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   - Perturbation theory concept
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Differential Equations: The Basics

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A Simple Example: Population Modeling

Population growth is commonly modeled with differential equations. In the following equation: $t =$ time, $P =$ population and $k =$ proportionality constant. k represents the constant ratio between the growth rate of the population and the size of the population.

$$\frac{dP}{dt} = kP$$

In this particular equation, the left hand side represents the growth rate of the population being proportional to the size of the population P. This is a very simple example of a first order, ordinary differential equation. The equation only contains first order derivatives and there are no partial derivatives.

Initial Value Problems

An initial value problem consists of a differential equation and an initial condition. So, going back to the population example, the following is an example of an initial value problem:

$$\frac{dP}{dt} = kP, P(0) = P_0$$

The solution to this set of equations is a function, call it $P(t)$, that satisfies both equations.

Linear First Order Differential Equations

- The standard form for a first-order differential equation is

$$\frac{dy}{dt} = f(t, y)$$

where the right hand side represents the function f that depends on the independent variable, t, and the dependent variable, y.

General Solutions to a Differential Equation

Let's look at a simple example and walk through the steps of finding a general solution to the following equation

$$\frac{dy}{dt} = (ty)^2$$

We will simply "separate" write as "separate" the variables then integrate the both sides of the equation to find the general solution.

$$\begin{aligned} \frac{dy}{dt} &= t^2 y^2 \\ \frac{1}{y^2}\, dy &= t^2\, dt \\ \int \frac{1}{y^2}\, dy &= \int t^2\, dt \end{aligned}$$

$$
\begin{aligned}
-y^{-1} &= \frac{t^3}{3} + c \\
-\frac{1}{y} &= \frac{t^3}{3} + c \\
y &= -\frac{1}{\frac{t^3}{3} + c} \\
\Rightarrow y(t) &= -\frac{3}{t^3 + c_1}
\end{aligned}
$$

where c_1 is any real number.

Linear First Order Differential Equations

Initial value problems consist of a differential equation and an initial value. We will work through the example below:

$$\frac{dx}{dt} = -xt; \qquad x(0) = \frac{1}{\sqrt{\pi}}$$

First we will need to find the general solution to $\frac{dx}{dt} = -xt$, then use the initial value $x(0) = \frac{1}{\sqrt{\pi}}$ to solve for c. Since we do not know what $x(t)$ is, we will need to "separate" the equation before integrating.

$$\begin{aligned} \frac{dx}{dt} &= -xt \\ -\frac{1}{x}\,dx &= t\,dt \\ \int -\frac{1}{x}\,dx &= \int t\,dt \end{aligned}$$

Linear First Order Differential Equations Continued

$$
\begin{aligned}
-\ln x &= \frac{t^2}{2} + c \\
x &= e^{-(\frac{t^2}{2}+c)} \\
x &= e^{-(\frac{t^2}{2})} e^{-c} \\
x &= ke^{-\frac{t^2}{2}}
\end{aligned}
$$

The above function of t is the general solution to $\frac{dx}{dt} = -xt$ where k is some constant. Since we have the initial value $x(0) = \frac{1}{\sqrt{\pi}}$, we can solve for k.

Solving Initial Value Problems

Thus we can see that the solution to the initial value problem

$$\frac{dx}{dt} = -xt; x(0) = \frac{1}{\sqrt{\pi}}$$

is

$$x(0) = \frac{1}{\sqrt{\pi}} = ke^{-\frac{0^2}{2}}$$

$$x(t) = \frac{1}{\sqrt{\pi}}e^{-\frac{t^2}{2}}$$

Let's verify that this solution is correct. We will need to show

$$\begin{aligned}
\frac{dx}{dt} &= x'(t) = f(t, x(t)) \\
\frac{dx}{dt} &= \frac{d}{dt}\left(\frac{1}{\sqrt{\pi}} e^{-\frac{t^2}{2}}\right) = \frac{1}{\sqrt{\pi}} e^{-\frac{t^2}{2}} \\
&\Rightarrow \frac{d}{dt}\left(\frac{1}{\sqrt{\pi}} e^{-\frac{t^2}{2}}\right) = -\frac{1}{\sqrt{\pi}} e^{-\frac{t^2}{2}}
\end{aligned}$$

Second Order Differential Equations

Second order differential equations simply have a second derivative of the dependent variable. The following is a common example that models a simple harmonic oscillator:

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

where m and k are determined by the mass and spring involved. This second order differential equation can be rewritten as the following first order differential equation:

$$\frac{dv}{dt} = -\frac{k}{m}y$$

where v denotes velocity.

Second Order Differential Equations Continued

Referring back to some calculus knowledge, if $v(t)$ is velocity, then $v = \frac{dy}{dt}$. Thus, we can substitute in $\frac{dv}{dt}$ into our second order differential equation and essentially turn it into a first order differential equation.

$$\frac{d^2y}{dt^2} = -\frac{k}{m}y \Leftrightarrow \frac{dv}{dt} = -\frac{k}{m}y$$

Now we have the following system of first order differential equations to describe the original second order differential equation:

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -\frac{k}{m}y \end{aligned}$$

Second Order Differential Equations Continued

Consider the following initial value problem:

$$\frac{d^2y}{dt^2} + y = 0$$

with $y(0) = 0$ and $y'(0) = v(0) = 1$. Let's show that $y(t) = \sin(t)$ is a solution. Let $v = \frac{dy}{dt}$, then we have the following system:

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -y \end{aligned}$$

Second Order Differential Equations Continued

$$\begin{aligned}
\frac{dy}{dt} &= \frac{d}{dt}\sin(t) = \cos(t) = v \\
\frac{dv}{dt} &= -\sin(t) = -y \\
\Rightarrow \frac{d^2y}{dt^2} &= -\sin(t) \\
\Rightarrow \frac{d^2y}{dt^2} + y &= \frac{d^2(\sin(t))}{dt^2} + \sin(t) \\
&= -\sin(t) + \sin(t) = 0
\end{aligned}$$

High Order Differential Equations as a System

Boundary Value Problems: The Basics

Power Series Solutions

To demonstrate how to use power series to solve a nonlinear differential equation we will look at Hermite's Equation:

$$\frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 2py = 0$$

We will use the following power series and its first and second derivatives to make a guess:

$$y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + ... = \sum_{n=0}^{\infty} a_n t^n \quad (1)$$

$$\frac{dy}{dt} = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + ... = \sum_{n=1}^{\infty} n a_n t^{n-1} \quad (2)$$

$$\frac{d^2y}{dt^2} = 2a_2 + 6a_3 t + 12a_4 t^2 + ... = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} \quad (3)$$

From the previous equations we can conclude that

$$\begin{aligned} y(0) &= a_0 \\ y'(0) &= a_1 \end{aligned}$$

Next we will substitute (1), (2) and (3) into Hermite's Equation and collect like terms.

$$\begin{aligned} \frac{d^2y}{dt^2} - 2t\frac{dy}{dt} + 2py = 0 = (2a_2 + 6a_3t + 12a_4t^2 + \ldots) \\ -2t(a_1 + 2a_2t + 3a_3t^2 + 4a_4t^3 + \ldots) \\ +2p(a_0 + a_1t + a_2t^2 + a_3t^3 + \ldots) \\ \Rightarrow (2pa_0 + 2a_2) + (2pa_1 - 2a_1 + 6a_3)t + \\ (2pa_2 - 4a_2 + 12a_4)t^2 + (2pa_3 - 6a_3 + 20a_5)t^3 = 0 \end{aligned}$$

Then from here, we will set all coefficients equal to 0 since the equation is equal to 0 and $t \neq 0$. We get the following sequence of equations:

$$\begin{aligned} 2pa_0 + 2a_2 &= 0 \\ 2pa_1 - 2a_1 + 6a_3 &= 0 \\ 2pa_2 - 4a_2 + 12a_4 &= 0 \\ 2pa_3 - 6a_3 + 20a_5 &= 0 \end{aligned}$$

Then will several substitutions we arrive at the following set of equations:

$$\begin{aligned}
\Rightarrow a_2 &= -pa_0 \\
a_3 &= -\frac{p-1}{3}a_1 \\
a_4 &= -\frac{p-2}{6}a_2 = \frac{(p-2)p}{6}a_0 \\
a_5 &= -\frac{p-3}{10}a_3 = \frac{(p-3)(p-1)}{30}a_1
\end{aligned}$$

Perturbation Theory Concept

Perturbation theory is used when a mathematical equation involves a small perturbation, usually ϵ. From here we create $y(x)$ such that it is an expansion in terms of ϵ. For example

$$y(x) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \cdots$$

This summation is called a perturbation series and it has a nice feature that allows each y_i to be solved using the previous y_i's. Consider the equation,

$$x^2 + x + 6\epsilon = 0, \quad \epsilon \ll 1 \tag{4}$$

Let's consider using perturbation theory to determine approximations for the roots of Equation (4).

Perturbation Theory Concept Continued

Notice this equation is a perturbation of $x^2 + x = 0$. Let $x(\epsilon) = \sum_{n=0}^{\infty} a_n \epsilon^n$. This series will be substituted into (4) and powers of ϵ will be collected. Next we will calculate the first term of the series by setting $\epsilon = 0$ in (4). So the leading order equation is

$$a_0^2 + a_0 = 0 \tag{5}$$

with solutions $x = -1, 0$. Thus $x(0) = a_0 = -1, 0$. Now the perturbation series are as follows

$$\begin{aligned} &= 1 - a_1\epsilon - a_2\epsilon^2 - a_1\epsilon + a_1^2\epsilon^2 + a_1a_2\epsilon^3 - a_2\epsilon^2 + a_1a_2\epsilon^3 + a_2^2\epsilon^4 - 1 + a_1\epsilon \\ &= (1-1) + (-2a_1 + a_1 + 6)\epsilon + (-2a_2 + a_1^2 + a_2)\epsilon^2 + \mathcal{O}(\epsilon^3) \end{aligned}$$

Perturbation Theory Concept Continued

$$ \tag{8} $$

$$x_1(\epsilon) = -1 + a_1\epsilon + a_2\epsilon^2 + \mathcal{O}(\epsilon^3) \tag{9}$$

and

$$x_2(\epsilon) \quad = \quad 0 + b_1\epsilon + b_2\epsilon^2 + \mathcal{O}(\epsilon^3) \tag{10}$$

Next, we will substitute in (9) into (4) while ignoring powers of ϵ greater than 2. Since we are only approximating the solution to the second-order, we can disregard the powers of ϵ greater than 2.

$$x^2 + x + 6\epsilon = (-1 + a_1\epsilon + a_2\epsilon^2)^2 + (-1 + a_1\epsilon + a_2\epsilon^2) + 6\epsilon$$
$$\Rightarrow (-a_1 + 6)\epsilon + (-a_2 + a_1^2)\epsilon^2 + \mathcal{O}(\epsilon^3)$$

Perturbation Theory Concept Continued

From here we take the coefficient of each power of ϵ and set it equal to zero. This step is justified because (4) is equal to zero and $\epsilon \neq 0$ so each coefficient must be equal to zero. Thus we have the following equations

$$\mathcal{O}(\epsilon^1) : -a_1 + 6 = 0$$
$$\mathcal{O}(\epsilon^2) : a_1^2 - a_2 = 0$$

These equations will be solved sequentially. The results are $a_1 = 6$ and $a_2 = 36$. Thus the perturbation expansion for the root $x_1 = -1$ is:

$$x_1(\epsilon) = -1 + 6\epsilon + 36\epsilon^2 + \mathcal{O}(\epsilon^3)$$

The same process can be repeated for x_2 with the perturbation expansion for the root $x_2 = 0$ resulting in

$$x_2(\epsilon) = -6\epsilon - 36\epsilon^2 + \mathcal{O}(\epsilon^3)$$

Concluding Remarks

Questions?